ISVAP Istituto per la Vigilanza sulle Assicurazioni Private e di Interesse Collettivo

Reserve Requirements and Capital Requirements in Non-Life Insurance

An analysis of the Italian MTPL insurance market by stochastic claims reserving models

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Introduction

In the framework of the *Solvency II Project*, the *Committee of European Insurance and Occupational Pensions Supervisors* (CEIOPS) has ben requested by the European Commission to establish well defined solvency and supervisory standards in order to allow a convergent and harmonized application across EU of the general prudential principles in the determination of the insurance technical provisions and the required solvency capitals.

It should be emphasized that a precise definition of technical provision allows the conceptual distinction between *reserve requirement* and *capital requirement* (or *risk capital*), which is a crucial point for deriving a consistent framework for insurance solvency. An important step in this direction is the new method proposed for computing technical provisions, which are defined by CEIOPS as the sum of two components:

- the *best estimate* component, representing the current expectation of the unpaid liabilities towards the policyholders,
- the *risk margin* component, expressing a prudential loading required to offset the liability uncertainty.

As concerning claims reserving in non-life insurance the relevant payoffs are the *Outstanding Loss Liabilities* (OLL), that is the future cash-flows representing claims generated by the policies written previously to the valuation date. Both the best estimate and the risk margin of the OLL must be derived by well defined probabilistic models properly calibrated on the relevant claims experience. However while the definition of best estimate as the (present value of) the expected OLL seems to be widely accepted, the question of how the risk margin of the OLL should be exactly defined is still under discussion. Two alternative approaches to risk margin definition are currently been considered:

- the *quantile* (or *percentile*) *approach*, which determines the technical provisions as a specified quantile (typically at 75% or 90% probability level) of the OLL probability distribution;
- the *cost-of-capital* (CoC) *approach*, which defines the risk margin as the cost of providing the solvency capital required to support the business-in-force until run-off.

At the end of 2005 a first round of *Quantitative Impact Study* (QIS1) across the Union Member States has been completed, mainly focused on the level of prudence to be embedded in the measurement of technical provisions under the risk margin approach. A second round of QIS (QIS2) is currently running, where the explicit calculation of solvency requirements for each risk driver in the insurance business is required to participants. Both the quantile approach and the CoC approach should be considered in the risk margin computation.

In this framework the Supervisor Authorities of some Union Member States (e.g. Germany, Portugal, Austria and Poland) have conducted exploratory analyses of the respective national insurance market, addressing quantitative issues, as the level of prudence embedded in current technical provisions, or facing qualitative questions, as the different methods used for determining risk margins.

In 2005 a study on the italian non-life insurance market has been started by ISVAP, aimed to verify the applicability of some stochastic loss reserving methods and to analyze their possible use in the supervisory activity. Following the current debate on the new reserving standards, the original scope of the study has been extended in order to include the valuation of the effects of the alternative definitions of best estimate, risk margin and solvency capital considered in the Solvency II framework. In particular, the analysis has been focused on the measurement of the level of prudence embedded in the italian *Motor Third-Party Liability* (MTPL) market, benchmarking the technical provisions set up in 2004 against alternative definitions of reserve and capital requirements. The risk margins have been computed under both the quantile and the CoC approach. The effects of discounting future liabilities have also been analysed, deriving best estimates and risk margins figures both on an undiscounted and on a discounted basis.

In this study the definitions of required reserve and required capital are developed in the theoretical framework of the fair valuation, prescribing that the future cash-flows are priced under the principle of market-consistency. As it is well-known, the economic theory assumes that the market price of a random cash-flow describing a future liability can be represented as the sum of the expected discounted cash-flow and a market risk premium, the latter expressing the loading currently required by the economic agents for facing the cash-flow uncertainty. Usually the price can be expressed as a risk-adjusted discounted expectation, that is by taking the expectation under a properly distorted probability distribution of the cash-flow, the so-called *risk-neutral* or *risk-adjusted* distribution. The separate specification of the "natural" expectation and of the risk premium is not required with this approach, since the proper risk loading is implicitly provided by the distortion of the probability measure. It is important to observe that the prices obtained in this way display the linearity property.

If the liability is not explicitly quoted on the market, but if a reference market however exists, the risk-neutral approach to fair valuation can be realized performing a "marked-to-model" pricing. The appropriate risk-adjusted distribution is derived in this case by calibrating a well-suited stochastic pricing model on the observed prices of analogous liability cash-flows; a market consistent price is then obtained by taking again the risk-neutral discounted expectation.

However fair valuation is not an easy task when non-life insurance liabilities are considered. Given that an efficient and well-developed reference market is not available in this case, a reliable risk-adjusted probability distribution cannot be identified and the only viable approach to valuation is to separately specify the natural expectation of the liability cash-flow and the corresponding risk premium. Since the risk premia are determined not only by the preferences (risk aversion) of the economic agents, but also by the uncertainty characterizing the liability cash-flows, also in this case a detailed stochastic model is needed in order to specify the probability distribution of the OLL. Market consistent risk loadings are then derived by taking into account any relevant information (also exogenously and indirectly obtained) concerning the preferences currently prevailing on the insurance market. Under this approach to fair valuation risk premia, hence prices, typically display a sub-additivity property; the linearity of prices can be assumed at most as an approximation.

The CEIOPS' definition of technical provision as the sum of best estimate and risk margin is clearly developed in this non-efficient market setting; by this point of view the alternative computations of risk margin proposed in the QIS2 framework appear as an attempt to gain a market-wide consensus on a possible measurement of the risk premia.

In order to apply the fair valuation principles in the non-efficient market setting, two alternative stochastic models for loss reserving have ben considered in this study, the Distribution-Free Chain-Ladder (DFCL) model, or "Mack's model", and the *Over-Dispersed Poisson* (ODP) model. Both models can be considered as a stochastic version of the classical *chain-ladder* method and require claims data in "triangular form", that is organized by accident year and development year. The ODP model has been applied by simulation, using a bootstrap procedure to describe the estimation uncertainty; this method provides the full probability distribution of the OLL. The DFCL model allows a closed form approach to the OLL uncertainty, but only produces the first two moments of the probability distribution; a full distribution has been obtained under the additional assumption of lognormality of the OLL.

The benchmarking of the observed technical provisions against alternative definitions of the required reserve and the measurement of the level of prudence embedded in the non-life insurance market in 2004 have been performed considering the MTPL segment and the connected line of business, the *Motor Kasko* (MK) segment. In term of statutory reserve these two lines of business represented about 59% of the overall italian non-life insurance market on December 31, 2004. More precisely, the analysis has been concerned with:

MTPL segment

- The reserve adequacy at the individual level, considering a selected sample of 40 companies, corresponding to 93% of the overall MTPL statutory reserves. These companies were selected from an original sample of 55 companies, after data from 15 companies containing incomplete or non-homogeneous informations have been discarded.
- The reserve adequacy on an aggregated basis, defining 4 dimensional classes determined by the amount of the statutory reserve. The total paid losses for each dimensional class, instead of the individual companies have been considered. In this part of the study we considered all the companies present on the MTPL market at the end of 2004, totalling 75 companies.

MK segment

- The reserve adequacy on an aggregate basis, considering the same dimensional classes determined for the MTPL segment.

Explicit assumptions have been made concerning the number of development years considered in the run-off triangles and the choice of the tail factors. As concerning the inflationary effects, two different approaches have been followed.

- In the first part of the analysis the loss reserving models have been applied directly to historical paid losses triangles. When applied to data expressed in terms of historical costs, the traditional run-off techniques for loss reserving implicitly assume the trend of past claims inflation as embedded into the cost development rule; therefore projected paid losses will include the inflation trend experienced in the past. The same effect will be observed under a stochastic loss reserving model, with the additional consequence that the variability of past inflation can influence also the variability of the predictive OLL distribution.
- In a second step a specific treatment of claims inflation has been performed. This is a relevant point, given that, as prescribed by CEIOPS [8], the inflation assumptions in the ultimate loss computation must be explicitly disclosed whenever the technical provisions

are derived on a discounted basis. In order to express the paid losses triangles in terms of current costs, a time series of inflation rates has been estimated on the claims inflation experienced in the MTPL segment; to this aim an apposite model has been used, suited for taking into account the estimated historical changes in the speed of finalization. Once the observed paid losses have been escalated by the estimated historical inflation, the probability distribution of the OLL has been derived using apposite uncertainty models for both technical cost development and claims inflation. These models were obtained by joining the stochastic loss reserving models without inflation with a lognormal model for stochastic inflation, where the future claims cost process is specified as a geometric brownian motion with given trend and volatility.

In the study the analysis of reserve adequacy has been developed considering both the two methods proposed in the framework of Solvency II for the computation of the risk margins, the quantile and the CoC method. Particular attention has been also devoted to the measurement of the *reserve risk* and of the corresponding capital requirement, the so-called *reserve risk capital*. A systematic comparison has been performed between risk capital figures obtained by the application of the DFCL and the ODP model to MTPL data, and the corresponding *Solvency Capital Requirement* (SCR) computed as specified in the QIS2 technical document.

In order to define a unified framework for the fair valuation and the risk control of non-life insurance liabilities a number of delicate methodological issues must be addressed. Some relevant problems tackled in the study are the following.

· As usual, risk capital measures are defined under a one-year view, requiring that solvency is guaranteed over a one year horizon and then iterating the solvency control at the beginning of each year. Hence in order to model the year-end obligations of the insurer the probability distribution of both the first year liabilities and of the year-end assessment of the residual reserve have been derived.
· Under a marked-to-model approach a rigorous risk capital assessment requires a "mixed approach" involving both risk-neutral and natural probabilities. The former are required for deriving the fair value at the year-end of the residual OLL, the latter are needed for computing its worst case assessment. Since a reliable fair valuation model is not readily available for non-life liabilities, some approximations must be made. A number of different approaches and approximations are considered and compared between each other in the study.
· It has been recognised that a CoC definition of risk margin can pose consistency problems with the corresponding risk capital definition[1]. A CoC approach to risk margin is proposed which avoids inconsistencies and seems economically meaningful.
· When liabilities with different maturities are aggregated on an undiscounted basis, the subadditivity property of risk margins produces a diversification effect of the overall risk. Of course this risk reduction effect must be saved when discounting is allowed; this requires a non-trivial application of the stochastic loss reserving models.

The rest of this paper is organised as follows. The first part is devoted to the theoretical framework. In the first chapter the general principles of fair valuation are illustrated and possible definitions of required reserve and required capital are considered. After a number

[1] See for example the "circularity" problems recently referred to by CRO Forum [10].

of reasonable approximations for the reserve risk capital have been introduced, corresponding consistent definitions of risk margin as CoC are provided.

In chapter two the classical run-off techniques for modelling non-life insurance liabilities are illustrated and the essential features of two stochastic models for claims reserving – the DFCL and the ODP model – are presented. For both models the problem of including the estimation error in the OLL variability is considered.

The applications to Italian market data are described in the second part of the paper. A description of the data used in the analysis is given in chapter 3. In chapter 4 the DFCL and the ODP model are applied to MTPL triangles of historical paid losses of individual companies, without any explicit treatment of claims inflation. In the first section the reserve adequacy is considered on an undiscounted basis. For each company in the selected sample and for the two models considered, risk margins as quantile, reserve risk capitals and risk margins as CoC are computed. A similar analysis is performed in the second section referring to discounted liabilities. An analysis of the MTPL and the MK market at aggregate level follows. Finally a detailed numerical illustration is given, comparing values of risk margins and risk capitals derived under alternative models and using different definitions and approximations, including the QIS2 specification of SCR.

In chapter 5 the reserve adequacy is studied considering the inflation effects. In the first section the adjustment of the historical payments for past inflation is considered. A time series of annual inflation rates for the MTPL segment is estimated considering historical market data on claims costs and claims counts. A simple model is used for correcting the inflation rates for changes in the speed of finalization (this model is described in appendix B). The estimated time series of inflation rates has ben used for escalating the historical paid losses, thus providing a triangle of inflation-adjusted payments for each company in the sample. The DFCL and the ODP model have been then applied to the inflation-adjusted triangles and the OLL probability distributions obtained have been compared with the corresponding distributions derived in chapter 4. This allows to appreciate the effects of the "embedded inflation" which is implicitly assumed using unadjusted triangles as payments data. In the second section the DFCL and the ODP model are "extended", including as an additional source of uncertainty a stochastic claims cost process, which is modelled as a geometric Brownian motion. The effects of the projected stochastic inflation are examined applying the extended models to the inflation-adjusted triangles.

Part I

The theoretical framework

Chapter 1

Reserve and capital requirements for non-life insurance liabilities

1.1 Reserve and capital requirements in an elementary setting

1.1.1 Best estimate, risk margin and fair value of the liabilities

At time t let us consider a specified line of business in P&C insurance and let us denote by L the corresponding total *Outstanding Loss Liabilities* (OLL) of the insurer. Let us suppose for the moment that the liabilities L will be paid at a single future date T and that discounting effects can be ignored. Of course L is a r.v. at time t.

Undiscounted required reserve

In this elementary setting the *required reserve* (RR) is defined as the sum R^* of a *best estimate* $\overline{L}$ of the OLL and a (non-negative) *risk margin* δ:

$$R^* := \overline{L} + \delta\,. \tag{1.1}$$

Of course both the best estimate $\overline{L}$ and the risk margin δ must be properly defined.

Best estimate

The most appropriate interpretation of best estimate (BE) seems to define $\overline{L}$ as the expectation $\mathbf{E}(L)$ of the OLL. As a general principle $\mathbf{E}(L)$ is the mean of the probability distribution of the unpaid liabilities L. It is worthwhile to mention that in practical applications one get a *predicted value* $\widehat{L}$ of the OLL provided by a suitable statistical model estimated on observed data. Since data are usually considered as a random observation sample, also the estimate $\widehat{L}$ is a r.v. and the predictive distribution of $\widehat{L}$ must be considered as providing the probability distribution of the OLL[1].

Remark. In some cases the *median* of the distribution has been proposed as an alternative definition of best estimate of L. ■

[1] Although this is the formulation of the predictive process usually found in the literature, a more rigorous description of the inferential problem would be obtained in the language of Bayesian statistics.

Risk margin and market value margin

While an interpretation of $\overline{L}$ as the expectation $\mathbf{E}(L)$ seems to be widely accepted, the appropriate definition of risk margin is currently an open issue. It is a common opinion that the risk margin (RM) should represent a prudential margin required to offset the large uncertainty of the unpaid liabilities. However this margin should not be raised up to an extreme level of protection. A reasonable approach seems to assume the required reserve R^* as a *market consistent* assessment of the debt of the insurer towards the policyholders, that is as the amount a well informed third party would require in a competitive market for relieving the insurer from its obligations. This point of view is declaratedly in line with the well-known definition of the *fair value* V of the unpaid liabilities. We shall denote by $\mathcal{V}(t; X_T)$ the market price at time t of the random amount to be paid at time T. Hence the fair value of the OLL is given by:

$$V = \mathcal{V}(t; L) .$$

Under the *fair value* (FV) *assumption* the required reserve is defined as the fair value of the liabilities:

$$R^* = \mathcal{V}(t; L) . \tag{1.2}$$

Hence R^* is the market price V of the random payoff L and the risk margin remains implicitly defined as the *risk loading* λ over the expected value $\mathbf{E}(L)$ required on the market by risk adverse investors in order to take over the liabilities L. This loading can also referred to as the *Market Value Margin.*

Under a more conservative point of view the risk margin δ could include an additional loading ε allowing for estimation uncertainty or for model uncertainty[2]. Thus the general definition of risk margin could be:

$$\delta := \lambda + \varepsilon , \tag{1.3}$$

where λ provides the Market Value Margin; the extra loading ε is equal to zero under the FV assumption.

Remark. Expression (1.3) with $\varepsilon = 0$ seems the interpretation accepted in the Swiss Solvency Test (see e.g.[23]). ■

Remark. When the required reserve is greater than the fair value, the positive extra loading $\varepsilon = R^* - V$ represents the time t value of future profits emerging during the life of the outstanding policy portfolio. It is usually referred to as the "value of business in force" which is the most important component of the so-called "embedded value" of the portfolio. ■

Remark. The previous definitions are referred to P&C insurance only to simplify the exposition. Similar definitions can be introduced also in life insurance. ■

Required reserve and certainty equivalent

Under the FV assumption the required reserve R^* can be considered as the market assessment of the *certainty equivalent* $\overline{\overline{L}}$ of the random amount L. The concept of certainty equivalent

[2] The idea of explicitly correcting a price assessment for model uncertainty has been recently introduced in option pricing by R. Cont [11].

is well-known in the traditional expected utility theory. In this setting expression (1.1) is the celebrated representation of the insurance premium as the sum of the pure premium and the appropriate security loading, where this loading is determined in a competitive market.

In an alternative framework the market certainty equivalent R^* can be represented as an expected value $\mathbf{E}^*(L)$ where the expectation is taken under an appropriate *distorted* (i.e. risk-adjusted) probability measure (see e.g. [39], [40]).

1.1.2 Risk-adjusted value of the liabilities and reserve risk capital

With the previous risk margin definition the required reserve is obviously not sufficient to meet usual solvency standards. In order to guarantee solvency at a given security level a *Risk Adjusted Value* (RAV) of the OLL is usually specified[3]. The RAV of L is intended as a very conservative valuation $\mathbf{W}(L)$ (a "worst case value") of the unpaid liabilities; for example $\mathbf{W}(L)$ can be fixed as the value of L which will not be exceeded with a fixed, very high level of probability (confidence level, security level). Hence the following inequalities hold:

$$\overline{L} < R^* < \mathbf{W}(L)\,.$$

Once the solvency level has ben specified by the RAV, the solvency capital required to the insurer is the amount K defined as:

$$K := \mathbf{W}(L) - R^* = \mathbf{W}(L) - \overline{L} - \delta\,. \tag{1.4}$$

This amount is usually referred to as *reserve risk (based) capital* (RC), at the specified security level. Other components of the solvency capital are usually considered, referred to additional risk drivers. The *premium risk capital*, for example, is related to the risk that losses for future claims are higher than premiums received. As only problems of reserve adequacy will be considered in the sequel, we shall refer to the reserve risk capital K simply as *risk capital* (RC). In the Solvency 2 documents the risk capital K is usually referred to as *Solvency Capital Requirement* (SCR) for reserve risk.

Remark. In definition (1.4) we implicitly assume that the amount R^* is exactly available to the insurer in order to meet the balance sheet constraint. In principle, if an additional capital (a "free surplus") F is available to the insurer, this amount can be subtracted from K. However this netting activity will be more properly managed in a final step, where the overall capital requirement for both assets and liabilities will be computed. ∎

Referring to a general random amount X, once the RAV $\mathbf{W}(X)$ has been specified it is useful to define the *unanticipated value of* X as the difference:

$$\mathbf{U}(X) := \mathbf{W}(X) - \mathbf{E}(X)\,.$$

With this definition expression (1.4) reads:

$$K = \mathbf{U}(L) - \delta\,, \tag{1.5}$$

which identifies the risk capital as the unanticipated loss (referred to the specified RAV) minus the risk margin.

[3] The RAV can be thought of as being fixed by the market regulator. In many cases however $\mathbf{W}(L)$ is a target level chosen by the insurance company in order to attain a fixed credit rating.

Remark. The expectation $\mathbf{E}(L)$ is also referred to as *anticipated loss*. Hence the required reserve R^* is given by the anticipated loss plus the risk margin. ■

In this very simple setting the RAV of the OLL required by the regulator to the insurer is decomposed into the sum of the required reserve R^* and of the risk capital K. It is worthwhile to observe that (at least under the FV assumption) the first component represents the market price of the OLL and, in principle, should be provided by the policyholders (i.e. covered by premiums). The second component is provided by the insurer (i.e. by the shareholders) and with high probability will result unused after the liabilities have been paid; however a cost for interest has to be considered for this capital since an appropriate return will be required by the shareholders.

The cost of risk capital

Let us denote by h the rate of return required at time t by the shareholders for investing in the risky asset represented by the insurance business we are considering. If i is the risk-free rate of return prevailing on the market, the cost at time t of the reserve risk capital (again ignoring discounting) is given by:

$$\kappa := s\,K\,,$$

where $s := h - i$ is the spread between the shareholders' return and the risk-free rate.

1.1.3 Risk margin as the cost of risk capital

In an efficient security market the excess return s is non-negative and represents the *risk premium* required in equilibrium by the investors for holding the insurance security. Hence it could make sense to use the cost of the risk capital κ as a proxy of the risk margin δ. Posing $\delta = \kappa = s\,K$ in (1.5) one has $K = \mathbf{U}(L) - s\,K$; hence one obtains:

$$K = \frac{\mathbf{W}(L) - \overline{L}}{1+s}\,, \tag{1.6}$$

and:

$$\delta = \frac{s}{1+s}\left[\mathbf{W}(L) - \overline{L}\right]. \tag{1.7}$$

Since $R^* = \overline{L} + \delta$ the required reserve can be expressed as:

$$R^* = \frac{\overline{L} + s\,\mathbf{W}(L)}{1+s}\,. \tag{1.8}$$

With the cost-of-capital approach the reserve problem and the capital problem are then unified into a single problem. Both the risk capital and the market value margin are simultaneously determined by expression (1.6) and (1.7) once the RAV $\mathbf{W}(L)$ and the spread s have been specified. Under our simplified assumptions the required reserve is a weighted average of the best estimate $\overline{L}$ and of the RAV of the liabilities, with weight $1/(1+s)$ and $s/(1+s)$, respectively.

1.1.4 Examples

Assuming $\overline{L} := \mathbf{E}(L)$ as the best estimate of L, on can consider some relevant cases. We refer to section A in the Appendix for an explicit definition of notations.

a) *No risk margin, RAV as a quantile.*

$$\delta = 0\,, \qquad \mathbf{W}(L) = \mathbf{Q}^{(99.9)}(L)\,,$$

where $\mathbf{Q}^{(\alpha)}(L)$ is the α-th quantile of L. Hence one has:

$$R^* = \mathbf{E}(L)\,, \qquad K = \mathbf{Q}^{(99.9)}(L) - \mathbf{E}(L)\,.$$

The required reserve is given by the *expected losses* and the risk capital is equal to the *unexpected losses* (at a 99.9% confidence level).

b) *Risk margin and RAV as a quantile.*

$$\delta = \mathbf{Q}^{(75)}(L) - \mathbf{E}(L)\,, \qquad \mathbf{W}(L) = \mathbf{Q}^{(99.9)}(L)\,.$$

Hence one has:

$$R^* = \mathbf{Q}^{(75)}(L)\,, \qquad K = \mathbf{Q}^{(99.9)}(L) - \mathbf{Q}^{(75)}(L)\,.$$

c) *Risk margin and RAV as σ-affine functions.*

$$\delta = \eta'\,\mathbf{Std}(L)\,, \qquad \mathbf{W}(L) = \mathbf{E}(L) + \eta''\,\mathbf{Std}(L)\,, \quad \eta'' > \eta' > 0\,,$$

where $\mathbf{Std}(L)$ is the standard deviation of L. In this case:

$$R^* = \mathbf{E}(L) + \eta'\,\mathbf{Std}(L)\,, \qquad K = (\eta'' - \eta')\,\mathbf{Std}(L)\,.$$

d) *Risk margin as the cost of capital, RAV as a quantile.*

$$\delta = \kappa = s\,K\,, \qquad \mathbf{W}(L) = \mathbf{Q}^{(99.9)}(L)\,,$$

which implies:

$$R^* = \frac{\mathbf{E}(L) + s\,\mathbf{Q}^{(99.9)}(L)}{1+s}\,, \qquad K = \frac{\mathbf{Q}^{(99.9)}(L) - \mathbf{E}(L)}{1+s}\,.$$

e) *Risk margin as the cost of capital, RAV as the expected shortfall.*

$$\delta = \kappa = s\,K\,, \qquad \mathbf{W}(L) = \mathbf{S}^{(99.9)}(L)\,,$$

where $\mathbf{S}^{(\alpha)} := \mathbf{E}\big(L | L \geq \mathbf{Q}^{(\alpha)}\big)$ is the *expected shortfall* at the level α, that is the expected loss beyond the α-th quantile (also referred to as *Tail-VaR*). Therefore one has:

$$R^* = \frac{\mathbf{E}(L) + s\,\mathbf{S}^{(99.9)}(L)}{1+s}\,, \qquad K = \frac{\mathbf{S}^{(99.9)}(L) - \mathbf{E}(L)}{1+s}\,.$$

1.2 Reserve and capital requirements under maturity structure of the liabilities

Typical insurance liabilities have a complex maturity structure ranging over a many-year time horizon. Also in non-life insurance the liabilities of long-tailed lines of business have over-ten-years maturities. This maturity structure and the relative discounting effects must be properly taken into account both in the reserve and in the risk capital definition problem.

At time $t = 0$ let us consider the outstanding liabilities generated by the P&C policies collected in the previous years. These OLL will be represented by a stream:

$$\boldsymbol{Y} := \left\{Y_\tau;\ \tau = 1, 2, \ldots, T\right\},$$

of random payoffs ranging over T years from now, the payoff Y_τ being due at the end of year τ. Under the FV assumption the required reserve at time $t = 0$ is given by $R_0^* = V_0$, where:

$$V_0 := \mathcal{V}(0; \boldsymbol{Y}),$$

is the fair value of the liability stream.

For homogeneity with the previous notations we shall denote by

$$L := \sum_{\tau=1}^{T} Y_\tau,$$

the sum of the OLL.

1.2.1 Fair value of the liabilities

The theory of efficient financial markets provides a formal definition of the fair value of a payment stream. At time t and for $\theta \geq 0$ let us denote by $v(t, t+\theta)$ the market price at time t of the default-free unit zero-coupon bond (ZCB) with maturity $t + \theta$, that is the time t value of one unit of money to be paid with certainty after θ units of time. By definition:

$$v(t, t+\theta) := [1 + i(t, t+\theta)]^{-\theta},$$

where $i(t, t+\theta)$ is the risk-free interest rate prevailing on the market at time t for the maturity $t + \theta$. Of course $v(t, t) = 1$.

For the simplicity sake we assume that there is no uncertainty on the future interest rates, hence the future prices $v(t, t+\theta)$ are assumed to be known at time zero.

Remark. In many non-life insurance applications the assumption of certain interest rates can provide reasonable approximations of the valuation problem. However the assumption can be relaxed adopting a more complex formalization. ■

At time $\tau = 0, 1, \ldots, T,$ let us denote by:

$$\boldsymbol{Y}^{(\tau)} := \left\{Y_\theta;\ \theta = \tau+1, \tau+2, \ldots, T\right\},$$

the residual liability stream, that is the stream of the liabilities still outstanding at the end of the year τ. Of course $\boldsymbol{Y}^{(0)} = \boldsymbol{Y}$ and $\boldsymbol{Y}^{(T)} = 0$. Under the arbitrage principle in perfect markets the fair value of the OLL at time τ has the following expression[4]:

$$V_\tau := \mathcal{V}\big(\tau; \boldsymbol{Y}^{(\tau)}\big) = \sum_{\theta=\tau+1}^{T} v(\tau,\theta)\, \mathbf{E}_\tau^Q(Y_\theta)\,, \tag{1.9}$$

where $\mathbf{E}_\tau^Q$ is the expectation at time τ taken with respect to the *risk-neutral* probability measure $\mathbf{Q}$. This expectation provides, by definition, the market risk loading over the "natural" expectation $\mathbf{E}_\tau(Y_\theta)$; hence $\mathbf{E}_\tau^Q(Y_\theta)$ can be interpreted as the certainty equivalent fixed at time τ on the market for the random liability Y_θ; the corresponding market risk loading is given by:

$$\gamma(\tau,\theta) := \mathbf{E}_\tau^Q(Y_\theta) - \mathbf{E}_\tau(Y_\theta)\,, \tag{1.10}$$

where $\mathbf{E}_\tau$ is the expectation at time τ taken with respect to the *natural* probability measure. Since the agents on the market are assumed to be risk-averse $\gamma(\tau,\theta)$ is non-negative.

Representation (1.9) implies the linearity property of the valuation functional $\mathcal{V}$. In particular one has:

$$\mathcal{V}\big(\tau; \boldsymbol{Y}^{(\tau)}\big) = \sum_{\theta=\tau+1}^{T} \mathcal{V}(\tau; Y_\theta)\,, \tag{1.11}$$

where:

$$\mathcal{V}(\tau; Y_\theta) := v(\tau,\theta)\, \mathbf{E}_\tau^Q(Y_\theta)\,, \tag{1.12}$$

is the fair value of the individual liability Y_θ. At time zero we have:

$$V_0 := \mathcal{V}(0; \boldsymbol{Y}) = \sum_{\tau=1}^{T} v_\tau\, \overline{\overline{Y}}_\tau\,, \tag{1.13}$$

where we used the simplified notation $v_\tau := v(0,\tau)$ and we denoted by:

$$\overline{\overline{Y}}_\tau := \mathbf{E}_\tau^Q(Y_\tau)\,,$$

the time zero certainty equivalent of Y_τ.

We also denote by V_τ^- the "cum dividend" value:

$$V_\tau^- := Y_\tau + V_\tau\,,$$

that is the fair value of the OLL immediately before the current liability Y_τ has been paid. The important property holds[5]:

$$v_1\, \mathbf{E}_0^Q\big(V_1^-\big) = V_0\,. \tag{1.14}$$

[4]This relation corresponds to the fundamental martingale property in asset pricing. It can be shown (see e.g. [17], pp. 22-29) that the "discounted gain process" defined as:

$$\widetilde{V}_\tau := v(0,\tau)\, V_\tau + \sum_{k=1}^{\tau} v(0,k)\, Y_k\,, \quad \tau = 0,1,\ldots,T\,,$$

is a martingale with respect to the risk-neutral measure; that is:

$$\mathbf{E}_\tau^Q(\widetilde{V}_\theta) = \widetilde{V}_\tau\,, \quad 0 \le \tau \le \theta \le T\,;$$

the price representation (1.9) is obtained for $\theta = T$.

[5]By the martingale property $\mathbf{E}_\tau^Q(\widetilde{V}_{\tau+1}) = \widetilde{V}_\tau$; in particular, for $\tau = 0$ one has $V_0 = \mathbf{E}_0^Q(\widetilde{V}_1)$, that is:

$$V_0 = \mathbf{E}_0^Q(v_1\, V_1 + v_1\, Y_1) = v_1\, \mathbf{E}_0^Q(V_1^-)\,.$$

1.2.2 Expected present value of the liabilities

The expected present value of the OLL at time τ is defined as:

$$M_\tau := \sum_{\theta=\tau+1}^{T} v(\tau,\theta)\, \mathbf{E}_\tau(Y_\theta)\,.$$

By the arbitrage principle, under deterministic interest rates this is equivalent to:

$$M_\tau := \sum_{\theta=\tau+1}^{T} v(0,\tau,\theta)\, \mathbf{E}_\tau(Y_\theta)\,. \tag{1.15}$$

where:

$$v(0,\tau,\theta) := \frac{v_\theta}{v_\tau}\,,$$

is the time zero forward rate from τ to θ (see e.g. [5], pp. 340-341). One can interpret M_τ as the *discounted best estimate* of the OLL at time τ.

At time zero one has:

$$M_0 = \sum_{\tau=1}^{T} v_\tau\, \overline{Y}_\tau\,, \tag{1.16}$$

where $\overline{Y}_\tau$ denotes the expectation $\mathbf{E}_0(Y_\tau)$ of Y_τ.

The "cum dividend" discounted expectation is given by:

$$M_\tau^- := Y_\tau + M_\tau\,, \tag{1.17}$$

and the following property is immediately proven:

$$v_1\, \mathbf{E}_0\big(M_1^-\big) = M_0\,. \tag{1.18}$$

1.2.3 The reserve as the fair value of the liabilities

Under the FV assumption the required reserve R_τ^* of the OLL at time τ is the fair value of the residual liability stream:

$$R_\tau^* := V_\tau = \mathcal{V}\big(\tau; \mathbf{Y}^{(\tau)}\big)\,, \quad \tau = 0, 1, \ldots, T\,. \tag{1.19}$$

The corresponding market value margin is defined as:

$$\begin{aligned}\lambda_\tau := V_\tau - M_\tau &= \sum_{\theta=\tau+1}^{T} v(\tau,\theta)\, \big[\mathbf{E}_\tau^Q(Y_\theta) - \mathbf{E}_\tau(Y_\theta)\big] \\ &= \sum_{\theta=\tau+1}^{T} v(\tau,\theta)\, \gamma(\tau,\theta)\,.\end{aligned} \tag{1.20}$$

At time zero:

$$\lambda_0 = V_0 - M_0 = \sum_{\tau=1}^{T} v_\tau\, \gamma_\tau\,, \tag{1.21}$$

where $\gamma_\tau := \gamma(0,\tau)$.

1.2.4 Special cases

It is worth considering some special cases of the previous general setting obtained introducing some simplifications useful for practical applications or for illustration purposes.

Undiscounted case

This is the case just considered in the introductory section 1.1. It is obtained posing $v(t, t+\theta) = 1$ for all t and θ, that is assuming a term structure of interest rates deterministic and flat at zero level. In this case one can ignore the maturity structure of the OLL and can imagine the total liabilities $L = \sum_{\tau=1}^{T} Y_\tau$ as falling due immediately after the valuation date.

Flat case

The flat case is the undiscounted case without risk margins, that is also assuming $\gamma(\tau, \theta) = 0$ for all τ and θ. In this case the risk-neutral measure coincides with the natural measure and the required reserve R^* is given by the best estimate $\overline{L}$ of the total liabilities.

The flat case describes the traditional setting of P&C loss reserving, where one usually assumes that the undiscounted valuation of the liabilities implicitly includes the appropriate risk loadings. While being extremely simplified by the financial point of view, this approach maintains a great deal of complexity concerning the probabilistic characterization of the OLL. Most of the stochastic models for loss reserving are defined in this framework.

Single-maturity case

The case with a single one-year maturity obtained for $T = 1$ is often considered, usually for illustration purposes. In this case both the discounting effect, given by v_1, and the risk margin $\lambda_0 = v_1 \gamma_1$ can be taken into account. However this single maturity simplification conceals much of the complexity of the reserving process and of the related solvency problem.

1.2.5 Reserve risk capital under the one-year view

As the reserve requirement is maintained until the run-off of the OLL, also the solvency requirement, i.e. the risk capital, must be set up in order to guarantee the insurer's solvability until the terminal liability Y_T has been paid. However in solvency regulation a one-year view is usually adopted defining at time zero the RAV W_0 with respect to the insurer's obligations at the end of the next accounting year and then iterating the procedure over the whole run-off of the policy portfolio. Under this kind of recursive procedure a sequence of one-year risk capitals $K_0, K_1, \ldots, K_{T-1}$ will be defined, where K_τ is the risk capital required at time τ for the year ending at time $\tau + 1$.

A widely adopted RAV definition is obtained specifying W_0 as an α-quantile $\mathbf{Q}_0^\alpha(V_1^-)$ at a very high confidence level (e.g. $\alpha = 99.9\%$). Alternative definitions are obtained specifying the RAV as a coherent risk measure (see [2]). For example one can choose W_0 as the expected shortfall $\mathbf{S}_0^\alpha(V_1^-)$ at the α confidence level. For a generic random amount X, we shall assume that the RAV operator $\mathbf{W}_0(X)$ satisfies at least the following properties:

- *Positive homogeneity*: $\mathbf{W}_0(c\,X) = c\,\mathbf{W}_0(X)$ for all constant $c \geq 0$;
- *Translation invariance*: $\mathbf{W}_0(c + X) = c + \mathbf{W}_0(X)$ for all constant c.

Remark. The RAV definition $\mathbf{W}_0(X)$ can be considered as deriving from the risk measures $\rho(X)$ defined by [2] and could be rigorously characterized based on axiomatic properties. In the general axiomatic framework however the r.v. X usually represents the *net worth* of a financial position. In a general approach to insurance solvency X should be the *surplus* $S := A - V$ of the company, where V is the fair value of the overall outstanding liabilities and A is the market value of the assets backing these liabilities. We are interested here only in the liability side and we consider the capital requirement of a single line of business of the non-life activity. Of course a number of aggregation steps will be needed in order to tackle the overall solvency problem including diversification and compensation effects. We refer to [22] for a nice introduction to the insurance solvency problem in a general setting. ■

Definition of reserve risk capital

Let us consider the risk capital K_0 required at time $t = 0$ for the first year. The insurer's obligations at time $t = 1$ are the sum of two random variables: the liabilities Y_1 due for the current year and the reserve R_1^* required for the remaining stream of the outstanding liabilities. With our notations (and under the FV assumption) these obligations are given by the cum-dividend fair value V_1^-. Hence the RAV of the year-end obligations is:

$$W_0 := \mathbf{W}_0(V_1^-) = \mathbf{W}_0(Y_1 + V_1)\,. \tag{1.22}$$

In order to obtain this money amount at time $t = 1$ the sum $v_1\, W_0$ is required at time $t = 0$ to the insurer. If the sum $R_0^* = V_0$ is exactly available, the *reserve risk capital* is then given by:

$$K_0 := v_1\, \mathbf{W}_0(V_1^-) - V_0\,. \tag{1.23}$$

In the simple one-maturity case (i.e. for $T = 1$) one has $V_1 = 0$ hence $W_0 = \mathbf{W}_0(Y_1)$ and:

$$K_0 = v_1\, \mathbf{W}_0(Y_1) - V_0\,. \tag{1.24}$$

Therefore the reserve problem and the risk capital problem can be viewed as two separated issues. The former involves only the risk-neutral distribution of Y_1, the latter only requires the computation of the RAV of the natural distribution of Y_1.

In the multiperiod case the situation is more complex:

- the reserve problem (i.e. the fair valuation problem) involves the T-dimensional p.d. of the r.v. $\boldsymbol{Y}$ under the risk-adjusted measure;
- under the one-year approach, the risk capital problem (1.23) requires considering the univariate natural probability distribution of the future reserve V_1^-, which in turn is determined by the risk-adjusted probability at time $t = 1$.

Hence while natural expectations are – in principle – not needed for determining the reserve, the risk capital assessment requires a "mixed approach" involving both natural and risk-neutral probabilities. If a reliable market model for determining the fair value V_τ is available then the risk capital problem can be tackled with only slight additional difficulties. This is the typical situation when the financial component of life insurance portfolios is being valuated. However when a robust fair valuation model is not readily available it is difficult to derive a risk capital measure fully consistent with definition (1.23) and some approximations are needed.

Before considering possible approximations it is worthwhile to derive an alternative representation of the risk capital based on the definition of the unanticipated value $\mathbf{U}_0(X) :=$

$\mathbf{W}_0(X) - \mathbf{E}_0(X)$ just introduced in section 1.1.2. Recalling property (1.14), expression (1.23) can also be written as:

$$K_0 = \mathbf{U}_0(v_1 V_1^-) - \left[\mathbf{E}_0^Q(v_1 V_1^-) - \mathbf{E}_0(v_1 V_1^-)\right]. \tag{1.25}$$

This expression makes clear that the risk capital can be obtained as the unanticipated value of the discounted year-end obligations (computed on the natural p.d.), provided that a correction is made to take into account the corresponding market risk margin.

In the single-maturity case one has:

$$K_0 = \mathbf{U}_0(v_1 Y_1) - \lambda_0 , \tag{1.26}$$

where the market value margin is now given by $\lambda_0 = v_1 \gamma_1$.

1.2.6 Approximated risk capital measures

Approximation by variability of year-end expectation

The previous definition of reserve risk capital is based on the variability of the year-end obligations, which are the sum of the current-year liability Y_1 and of the new reserve assessment V_1. The most natural approximation to expression (1.23) is probably obtained by substituting the variability of the year-end reserve with the variability of year-end (discounted) expectation of future liabilities.

a) YEE approximation. Using the definitions (1.13) and (1.10) the V_1^- value can be written as:

$$\begin{aligned} V_1^- &= Y_1 + \sum_{\tau=2}^{T} v(1,\tau)\left[\mathbf{E}_1(Y_\tau) + \gamma(1,\tau)\right] = M_1^- + \sum_{\tau=2}^{T} v(1,\tau)\,\gamma(1,\tau) \\ &= M_1^- + \frac{1}{v_1}\sum_{\tau=2}^{T} v_\tau\,\gamma(1,\tau) , \end{aligned}$$

where the last equality holds by the assumption of deterministic interest rates. We assume that in the RAV computation the market risk loadings at time $t = 1$ can be approximated by the current risk loadings for the corresponding maturities; that is:

$$\gamma(1,\tau) \approx \gamma(0,\tau) , \quad \tau = 2, 3, \ldots, T . \tag{1.27}$$

Hence, by the λ_0 definition (1.21):

$$\sum_{\tau=2}^{T} v_\tau\,\gamma(1,\tau) \approx \sum_{\tau=2}^{T} v_\tau\,\gamma_\tau = \lambda_0 - v_1\,\gamma_1 .$$

Therefore in expression (1.22) we substitute the r.v. V_1^- with the r.v.:

$$M_1^- + \lambda_0/v_1 - \gamma_1 .$$

obtaining the approximation:

$$\mathbf{W}_0(V_1^-) \approx \mathbf{W}_0(M_1^- + \lambda_0/v_1 - \gamma_1) . \tag{1.28}$$

We shall refer to this assumption as *the year-end expectation* (YEE) *approximation*. Of course, by the translation invariance we have $W_0 = \mathbf{W}_0(M_1^-) + \lambda_0/v_1 - \gamma_1$.

Using the RAV W_0 the YEE-approximated risk capital is defined as:

$$K_0 := v_1 \, W_0 - V_0 = v_1 \, \mathbf{W}_0(M_1^-) + \lambda_0 - v_1\gamma_1 - V_0 \, ,$$

that is, recalling that $V_0 = M_0 + \lambda_0$:

$$K_0 = v_1 \, \mathbf{W}_0(M_1^-) - M_0 - v_1\gamma_1 \, . \tag{1.29}$$

With the YEE approximation the solvency assessment is essentially reduced to a problem involving only the univariate natural distribution of the r.v. M_1^-. The RAV valuation also depends on the specification of the risk loading γ_1, but the modelling of the future risk loadings $\gamma(1,\tau)$ is not required.

In the single-maturity case one has $M_1^- = Y_1$ and $v_1\gamma_1 = \lambda_0$; hence:

$$K_0 = v_1 \, \mathbf{W}_0(Y_1^-) - (\overline{Y}_1 + \lambda_0) \, ,$$

which is equal to the exact expression of K_0 given by (1.24).

It is worthwhile to observe that in general the term $v_1\gamma_1$ is not negligible, since in typical P&C portfolios the liabilities Y_τ are decreasing with the maturity τ and the first-year payoff Y_1 can be of the same importance of the sum of the remaining payoffs.

Remark. The approximation (1.27) substitutes random with deterministic amounts, which tends to reduce the value of the RAV W_0. On the other hand the expectation at time $t = 0$ of the risk-loadings γ at time $t = 1$ should be lower than the current risk loading on the corresponding maturity since the time horizon is reduced by one year. Therefore in general it remains undetermined whether the YEE approximation implies an over- or an under-estimation of the RAV. ∎

As stated by (1.18), one has $\mathbf{E}_0(v_1 \, M_1^-) = M_0$. Then the YEE-approximated risk capital can be also expressed as:

$$K_0 = \mathbf{U}_0(v_1 \, M_1^-) - v_1\gamma_1 \, , \tag{1.30}$$

that is as the unanticipated value of the discounted year-end expectation minus the correction term $v_1\gamma_1$ which takes into account the risk loading of the first-year payoff not captured by the natural expectation. For $T = 1$ this relation is obviously the same as the exact expression (1.26).

b) φ-discounted YEE approximation. Let:

$$\overline{L} := \sum_{\tau=1}^{T} \overline{Y}_\tau \, ,$$

denote the total expected liabilities and let us define the *cumulative discount factor*:

$$\varphi := \frac{\sum_{\tau=1}^{T} v_\tau \, \overline{Y}_\tau}{\overline{L}} = \frac{M_0}{\overline{L}} \, . \tag{1.31}$$

The φ factor is the weighted average of the discount factors v_τ over the maturity range of the liabilities, the weights being the relative expected liabilities $\overline{Y}_\tau/\overline{L}$. One can say that φ captures the global discounting effect on the expected payoffs. Let us denote by:

$$Z_1 := Y_1 + \sum_{\tau=2}^{T} \mathbf{E}_1(Y_\tau)\,, \tag{1.32}$$

the undiscounted year-end expectation of the liability stream $\boldsymbol{Y}$. Of course $\mathbf{E}_0(Z_1) = \overline{L}$. The *$\varphi$-discounted YEE approximation* is defined by the assumption:

$$\mathbf{W}_0(v_1\, M_1^-) \approx \mathbf{W}_0(\varphi\, Z_1)\,. \tag{1.33}$$

With this approximation, given that $M_0 = \varphi\, \overline{L}$ by definition, expression (1.29) is changed in:

$$K_0 \approx \mathbf{W}_0(\varphi\, Z_1) - \varphi\, \overline{L} - v_1\gamma_1\,, \tag{1.34}$$

which can also be written as:

$$K_0 \approx \mathbf{U}_0(\varphi\, Z_1) - v_1\gamma_1\,.$$

c) Undiscounted YEE approximation. In the undiscounted case, i.e. for $v_\tau \equiv 1$, expression (1.29) or (1.34) reduces to:

$$K_0 \approx \mathbf{W}_0(Z_1) - \overline{L} - \gamma_1\,, \tag{1.35}$$

or $K_0 \approx \mathbf{U}_0(Z_1) - \gamma_1$, which corresponds to the traditional undiscounted approach largely used in P&C loss reserving.

Approximation by variability of future liabilities.

The p.d. of the year-end expectation M_1 of the residual liability stream $\boldsymbol{Y}^{(1)}$ required by the YEE approximation must be provided by a suitable stochastic model for the OLL. Many stochastic models for loss reserving provide the p.d. of the future liabilities Y_τ at maturity, but are not well suited for producing the p.d. of there expectation at time $t = 1$. This suggests further approximations.

a) LM approximation. Let us denote by:

$$D := \sum_{\tau=1}^{T} v_\tau\, Y_\tau\,, \tag{1.36}$$

the r.v. representing the sum of the discounted liabilities. Of course $\mathbf{E}_0(D) = M_0$. The *liability-at-maturity* (LM) *approximation* is defined by the assumption:

$$\mathbf{W}_0(v_1\, M_1^-) \approx \mathbf{W}_0(D)\,. \tag{1.37}$$

The LM-approximated risk capital is then given by:

$$K_0 \approx \mathbf{W}_0(D) - M_0 - v_1\gamma_1\,, \tag{1.38}$$

or:

$$K_0 \approx \mathbf{U}_0(D) - v_1\gamma_1\,.$$

b) φ-discounted LM approximation. Also in this case one can obtain a φ-discounted approximation using the cumulative discount factor $\varphi := M_0/\overline{L}$; that is assuming:

$$\mathbf{W}_0(D) \approx \mathbf{W}_0(\varphi\, L)\,. \tag{1.39}$$

This implies:

$$K_0 \approx \mathbf{W}_0(\varphi\, L) - \varphi\, \overline{L} - v_1\gamma_1\,, \tag{1.40}$$

or $K_0 \approx \mathbf{U}_0(\varphi\, L) - v_1\gamma_1$.

c) Undiscounted LM approximation. Under the undiscounted approach one has:

$$K_0 \approx \mathbf{W}_0(L) - \overline{L} - \gamma_1\,, \tag{1.41}$$

or $K_0 \approx \mathbf{U}_0(L) - \gamma_1$.

YEE and LM approximation in the flat case

As we just pointed out, most of the traditional approaches to loss reserving are defined in the flat case, that is they include neither discounting nor risk margins. In this case $R_0^* = \overline{L}$ and expressions (1.35) and (1.41) reduce, respectively, to:

$$K_0 \approx \mathbf{W}_0(Z_1) - \overline{L} = \mathbf{U}_0(Z_1)\,, \tag{1.42}$$

and:

$$K_0 \approx \mathbf{W}_0(L) - \overline{L} = \mathbf{U}_0(L)\,. \tag{1.43}$$

1.2.7 Subadditivity of risk margins

Under the FV assumptions the certainty equivalent of the liability Y_τ due at time τ has been defined by $\overline{\overline{Y}}_\tau := \mathbf{E}_0^Q(Y_\tau)$, that is as the expected payoff computed under the risk-neutral probability. Therefore the risk margin λ_0 defining the required reserve $R_0^* = M_0 + \lambda_0$ is the present value:

$$\lambda_0 = \sum_{\tau=1}^{T} v_\tau\, \gamma_\tau\,, \tag{1.44}$$

hence a linear combination of the individual risk loadings $\gamma_\tau = \overline{\overline{Y}}_\tau - \overline{Y}_\tau$.

However the risk-neutral measure can be unambiguously identified only if an efficient market for the liabilities $\boldsymbol{Y}$ exists. Given that for P&C liabilities this is typically not the case, risk loadings – and the corresponding certainty equivalents – have to be identified by suitable assumptions or making some approximations. As an additional problem, when the risk margins are not determined using risk-neutral expectations typical non-linearity problems naturally arise. In particular, the general definition of certainty equivalent in a multiperiod setting is of a delicate nature, involving the concept of intertemporal risk aversion (see for example [28], pp. 43-44).

Let us maintain the assumption of deterministic interest rates. At time $t = 0$ let us denote by $\overline{\overline{Y}}'_\tau$ the certainty equivalent of the liability Y_τ considered as separated from the others and assume now that $\overline{\overline{Y}}'_\tau$ is not determined as a risk-neutral expectation. We can write:

$$\overline{\overline{Y}}'_\tau := \overline{Y}_\tau + \gamma'_\tau\,,$$

where γ'_τ is a non-negative *stand-alone risk loading*, defined by some specified valuation principle. Correspondingly the required reserve is defined as:

$$R_0^* := M_0 + \lambda_0 \,,$$

where λ_0 is the positive risk margin provided by the same valuation principle. Referring to the r.v. D representing the sum of the discounted liabilities, we can also interpret R_0^* as the certainty equivalent $\overline{\overline{D}}$ (to be paid at time $t = 0$) of the total discounted liabilities D. However the following inequality usually holds:

$$\lambda_0 \leq \sum_{\tau=1}^{T} v_\tau \, \gamma'_\tau \,, \tag{1.45}$$

as a consequence of typical risk diversification effects. This subadditivity property is well-known in classical portfolio theory, where the risk margins are specified as σ-affine functions. But (1.45) is also valid for more general risk margin definitions, as those based on quantiles or expected shortfalls. Since the expectations are additive, relation (1.45) implies subadditivity for the discounted certainty equivalents:

$$\overline{\overline{D}} \leq \sum_{\tau=1}^{T} v_\tau \, \overline{\overline{Y}}'_\tau \,. \tag{1.46}$$

In order to recover additivity some kind of *allocated risk loading* γ_τ should be defined, characterized by the property (1.44). These risk loadings must be derived by specifying a conventional rule for allocating the total risk λ_0 to the single components Y_τ. Given the allocated risk loadings, the certainty equivalent of Y_τ *in the portfolio* is immediately defined as $\overline{\overline{Y}}_\tau := \overline{Y}_\tau + \gamma_\tau$ and the linearity property holds:

$$\overline{\overline{D}} = \sum_{\tau=1}^{T} v_\tau \, \overline{\overline{Y}}_\tau \,, \tag{1.47}$$

as with the risk-neutral measure approach.

Usually stochastic models for P&C liabilities provide assessments of both the stand-alone risk loadings γ'_τ and of the overall risk margin λ_0. Thus the correct value of the overall certainty equivalent $\overline{\overline{D}}$ can be readily computed, a measure of the diversification effect – if of interest – being provided by the difference $\overline{\overline{D}} - \sum \overline{\overline{Y}}'_\tau$. However typical actuarial models do not provide the allocated risk margins, which can be specified only under additional assumptions and/or by *ad-hoc* definitions.

As an example, for σ-affine risk margins the allocated risk loadings could be defined by the marginal contribution of the Y_τ component to the total risk margin λ_0. This approach only requires the specification of the covariance matrix between liabilities of different maturity, which is usually provided by actuarial stochastic models.

An important case is when the certainty equivalents are defined as α-quantiles. In this case the inequality (1.46) corresponds to:

$$\mathbf{Q}_0^{(\alpha)}(D) \leq \sum_{\tau=1}^{T} v_\tau \, \mathbf{Q}_0^{(\alpha)}(Y_\tau) \,. \tag{1.48}$$

where the quantiles $\mathbf{Q}_0^{(\alpha)}(Y_\tau)$ are computed on the p.d. of the individual liabilities Y_τ and the overall quantile $\mathbf{Q}_0^{(\alpha)}(D)$ is computed on the aggregate distribution of the discounted liabilities. The individual certainty equivalents $\overline{\overline{Y}}_\tau$ "in the portfolio" can be defined specifying the *allocation fractions*:

$$\beta_\tau := \frac{\overline{\overline{Y}}_\tau}{\mathbf{Q}_0^{(\alpha)}(D)}\,,$$

satisfying the property $\sum_{\tau=1}^{T} v_\tau\,\beta_\tau = 1$. Hence the allocated certainty equivalents $\overline{\overline{Y}}_\tau := \beta_\tau\,\mathbf{Q}_0^{(\alpha)}(D)$ satisfy (1.47) by definition; the corresponding allocated risk loadings are:

$$\gamma_\tau := \beta_\tau\,\mathbf{Q}_0^{(\alpha)}(D) - \overline{Y}_\tau\,. \tag{1.49}$$

For example, the fractions β_τ could be fixed as proportional to the expected liabilities:

$$\beta_\tau := \frac{\overline{Y}_\tau}{\sum_{\tau=1}^{T} v_\tau\,\overline{Y}_\tau}\,. \tag{1.50}$$

or to the stand alone quantiles:

$$\beta_\tau := \frac{\mathbf{Q}_0^{(\alpha)}(Y_\tau)}{\sum_{\tau=1}^{T} v_\tau\,\mathbf{Q}_0^{(\alpha)}(Y_\tau)}\,. \tag{1.51}$$

An alternative definition could be obtained specifying β_τ as the *covariance allocation fractions*:

$$\beta_\tau := \frac{\sum_{\theta=1}^{T} v_\tau v_\theta\,\mathbf{Cov}_0(Y_\tau, Y_\theta)}{\left[\sum_{k=1}^{T}\sum_{\theta=1}^{T} v_k v_\theta\,\mathbf{Cov}_0(Y_k, Y_\theta)\right]^{1/2}}\,. \tag{1.52}$$

The definition of the allocated risk loadings is also useful for deriving approximated reserve risk capitals, following the definitions provided in section 1.2.5. For example for quantile-based risk margins the basic approximation (1.29) gives, using (1.49):

$$K_0 = v_1\,\mathbf{W}_0(M_1^-) - M_0 - v_1\left[\beta_1\,\mathbf{Q}_0^{(\alpha)}(D) - \overline{Y}_1\right]. \tag{1.53}$$

1.3 Using reserve risk capital for defining risk margins

1.3.1 The cost of risk capital

Under the iterative approach based on the one-year view, once the principle determining the RAV has been chosen the corresponding reserve risk capital must be maintained in each year during the whole run-off of the portfolio. We denoted by K_τ ($\tau = 0, 1, \ldots, T-1$) the reserve risk capital for the year starting at time τ. For $\tau > 0$ the amount K_τ is a r.v. at time zero. Let:

$$\overline{K}_\tau := \mathbf{E}_0(K_\tau)\,, \quad \tau = 0, 1, \ldots, T-1\,,$$

be the time zero expectation of K_τ. Obviously $\overline{K}_0 = K_0$. Let h_τ be the *shareholders' return* in year $[\tau, \tau+1]$, that is the rate of return required by the shareholders for investing in the insurance business in the year starting at time τ. Moreover, let $i(\tau, \tau+1)$ be the one-year risk-free interest rate prevailing on the market at time τ. We define the *cost of risk capital* at time zero as:

$$\kappa_0 := \sum_{\tau=1}^{T} \mathcal{V}(0; c_\tau)\,,$$

where:

$$c_\tau := [h_\tau - i(\tau-1,\,\tau)]\,\overline{K}_{\tau-1}\,, \quad \tau = 1, 2, \ldots, T\,, \tag{1.54}$$

is the expected net interest amount to be paid by the insurer at the end of the year τ.

Usually one assumes $h_\tau \equiv h$ constant, known at time zero. Then it can be shown that the arbitrage principle requires:

$$\mathcal{V}(0; c_\tau) = v_\tau\,[h - i(0, \tau-1, \tau)]\,\overline{K}_{\tau-1}\,, \tag{1.55}$$

where $i(0, \tau-1, \tau)$ is the forward rate for the year τ which is implied by the current term structure of interest rates (see eg [5], par. 12.4). Hence one has:

$$\kappa_0 := \sum_{\tau=1}^{T} [h - i(0, \tau-1, \tau)]\, v_\tau\, \overline{K}_{\tau-1}\,. \tag{1.56}$$

It is worthwhile to observe that this property also holds under interest rate uncertainty.

A typical approximation for κ_0 is obtained assuming a deterministic and flat yield curve, posing:

$$i(\tau-1,\,\tau) \equiv i_1\,, \tag{1.57}$$

where i_1 is the current one-year risk-free interest rate. In this case one has $\mathcal{V}(0; c_\tau) = v_\tau\,(h - i_1)\,\overline{K}_{\tau-1}$.

More appropriately, also under interest rate uncertainty one can assume a constant spread:

$$s := h_\tau - i(\tau-1, \tau)\,, \quad \tau = 1, 2, \ldots, T\,.$$

Hence one has:

$$\kappa_0 = s \sum_{\tau=1}^{T} v_\tau\, \overline{K}_{\tau-1}\,. \tag{1.58}$$

Remark. In the Swiss Solvency Test a constant spread $s = 6\%$ is assumed. ■

The ongoing assumption

In many cases the expectations $\overline{K}_\tau$ of future risk capitals are not readily obtainable. Moreover under the run-off assumption the next-year obligations $V^-_{\tau+1}$ are vanishing with τ but can display increasing relative variability, thus providing unrealistic assessments of the risk capital. When the existing business is assumed to be effectively continued at a steady level it seems more realistic to susbstitute the expectations $\overline{K}_\tau$ for $\tau > 0$ in equations (1.54), (1.56) with:

$$\widehat{K}_\tau := q\,\widehat{M}_\tau\,, \tag{1.59}$$

where:

$$q := \frac{K_0}{M_0}\,,$$

and:

$$\widehat{M}_\tau := \sum_{\theta=\tau+1}^{T} v(0,\tau,\theta)\,\mathbf{E}_0(Y_\theta) = \frac{1}{v_\tau} \sum_{\theta=\tau+1}^{T} v_\theta\,\overline{Y}_\theta\,.$$

This expression provides an "ongoing" assessment of future risk capitals obtained by assuming in each year a constant proportion q between K_τ and the expectation of the corresponding residual reserve. The definition can be adopted also for $\tau = 0$, obviously assuming $\widehat{M}_0 = M_0$.

Under the *ongoing assumption* (1.59) on the future risk capitals and assuming a constant spread s, the interest amounts falling due at time τ are:

$$c_\tau = s\,q\,\widehat{M}_{\tau-1} = \frac{s\,q}{v_{\tau-1}} \sum_{\theta=\tau}^{T} v_\theta\,\overline{Y}_\theta\,, \quad \tau = 1,2,\ldots,T\,, \tag{1.60}$$

and the expression (1.58) assumes the simple form:

$$\kappa_0 = \sum_{\tau=1}^{T} v_\tau\,c_\tau = s\,q \sum_{\tau=1}^{T} \frac{v_\tau}{v_{\tau-1}} \sum_{\theta=\tau}^{T} v_\theta\,\overline{Y}_\theta\,. \tag{1.61}$$

1.3.2 Risk margins as the cost of risk capital

The cost of the reserve risk capital given by (1.61) can also be used for determining risk margins when indications from an efficient market are not available[6]. Since κ_0 can be interpreted as a kind of market consistent risk premium for the outstanding liability stream $\boldsymbol{Y}$, if a better risk margin assessment is not available one could assume κ_0 as the market value margin for the required reserve; that is:

$$\lambda_0 = \kappa_0\,. \tag{1.62}$$

Under the ongoing assumption the cost-of-capital approach has a simple and consistent formulation. Comparing (1.44) and (1.61) the following expression for the risk loadings must hold:

$$\gamma_\tau = c_\tau = s\,q\,\widehat{M}_{\tau-1}\,.$$

Of course the definition of risk capital K_0 and the definition of risk margin must be consistent. Under assumption (1.62) the first risk loading is given by:

$$\gamma_1 = c_1 = s\,K_0\,;$$

then if we adopt the risk capital representation given by the YEE approximation (1.29), we have:

$$K_0 = v_1\,\mathbf{W}_0(M_1^-) - M_0 - v_1\,s\,K_0\,,$$

[6] A cost-of-capital approach has been applied by the authors in March 2005 for determining the risk margins for technical risks in the life portfolios of a leading Italian insurance company. In this application the goal was a market consistent assessment of the "Value of Business In Force" derived by properly discounting best estimates of future profits provided by the outstanding policy portfolios. In this case the risk margins where then expressed in terms of *Risk Discount Margins*, i.e. as excess return over the risk-free rate. Details can be found in [7]. The cost of risk capital as a general rule for determining risk margins to be added to best estimates of liabilities has been also proposed in [33]. This definition seems to correspond quite closely to the definition of Market Value Margin for liabilities currently considered in the Swiss Solvency Test (see [23], [24], [29]). See also [25] for a comparison of different approaches to insurance liability valuation.

which gives:

$$K_0 = \frac{v_1\,\mathbf{W}_0(M_1^-) - M_0}{1 + v_1\,s} = \frac{\mathbf{U}_0(v_1\,M_1^-)}{1 + v_1\,s}\,. \tag{1.63}$$

Hence under assumption (1.62) subtracting $v_1\gamma_1$ from the unanticipated value of discounted year-end expectation M_1^- is the same as dividing it by $(1 + v_1\,s)$.

Correspondingly, the market value margin has the explicit expression:

$$\lambda_0 = s\,\frac{K_0}{M_0}\sum_{\tau=1}^{T} v_\tau\,\widehat{M}_{\tau-1}\,.$$

that is:

$$\lambda_0 = \frac{s}{1 + v_1\,s}\,\frac{v_1\,\mathbf{W}_0(M_1^-) - M_0}{M_0}\sum_{\tau=1}^{T}\frac{v_\tau}{v_{\tau-1}}\sum_{\theta=\tau}^{T} v_\theta\,\overline{Y}_\theta\,. \tag{1.64}$$

This relation expresses the risk margin as the product of three items:

$$\lambda_0 = \widehat{s}\,\widehat{u}\,\widehat{\mu}\,, \tag{1.65}$$

where:

- the "spread factor":

$$\widehat{s} := \frac{s}{1 + v_1\,s} = s\,\frac{1 + i_1}{1 + i_1 + s}\,,$$

  coinciding essentially with the spread s, provides an overall assessment of the market risk premium (in terms of excess return) for investments in the insurance business;
- the "u factor":

$$\widehat{u} := \frac{\mathbf{U}_0(M_1^-)}{\mathbf{E}_0(M_1^-)} = \frac{v_1\,\mathbf{W}_0(M_1^-) - M_0}{M_0}\,,$$

  gives the relative unanticipated year-end best estimate of the liabilities. It describes the intrinsic variability of the liability stream $\boldsymbol{Y}$, where the variability measure is conditional to the prudentiality level fixed by the RAV specification;
- the "liability factor":

$$\widehat{\mu} := \sum_{\tau=1}^{T} v_\tau\,\widehat{M}_{\tau-1}\,,$$

  is a money amount which captures the maturity structure of the expected liabilities, given the current term structure of interest rates.

Once the RAV $\mathbf{W}_0(M_1^-)$ has been computed, only the expected liabilities and the current risk-free discount factors are involved in expressions (1.63) and (1.64). The level of the interest rate spread s has not great importance in determining the risk capital, but is of strategic relevance for assessing the risk margin and thus the required reserve R_0^*.

The single-maturity case

If $T = 1$ one has:

$$M_1^- = Y_1\,,\quad M_0 = v_1\,\overline{Y}_1\,,\quad \lambda_0 = v_1\,c_1 = v_1\,s\,K_0\,;$$

hence:

$$K_0 = \frac{v_1 \left[\mathbf{W}_0(Y_1) - \overline{Y}_1\right]}{1 + v_1\, s}\,, \tag{1.66}$$

and:

$$\lambda_0 = \kappa_0 = s\, \frac{v_1^2 \left[\mathbf{W}_0(Y_1) - \overline{Y}_1\right]}{1 + v_1\, s}\,. \tag{1.67}$$

The required reserve is given by:

$$R_0^* = \frac{v_1}{1 + v_1\, s} \left[\overline{Y}_1 + v_1\, s\, \mathbf{W}_0(Y_1)\right], \tag{1.68}$$

It is interesting to observe that in this case one has:

$$\lambda_0 + K_0 = v_1\, \mathbf{U}_0(Y_1)\,,$$

that is the present value of the unanticipated loss is expressed as the sum of the market value margin and of the risk capital.

Other approximations

If the φ-discounted YEE approximation is adopted, expression (1.34) gives:

$$K_0 = \frac{\mathbf{W}_0(\varphi\, Z_1) - \varphi\, \overline{L}}{1 + v_1\, s} = \frac{\varphi\, \mathbf{U}_0(Z_1)}{1 + v_1\, s}\,, \tag{1.69}$$

and:

$$\lambda_0 = \frac{s}{1 + v_1\, s}\, \frac{\mathbf{U}_0(Z_1)}{\overline{L}} \sum_{\tau=1}^{T} v_\tau\, \widehat{M}_{\tau-1}\,. \tag{1.70}$$

Under the undiscounted YEE approximation (1.35) one has:

$$K_0 = \frac{\mathbf{U}_0(Z_1)}{1 + s}\,. \tag{1.71}$$

If the discount factors are set equal to 1 also in the discounted expectations $\widehat{M}_\tau$ one obtains:

$$\lambda_0 = \frac{s}{1 + s}\, \frac{\mathbf{U}_0(Z_1)}{\overline{L}} \sum_{\tau=1}^{T} \tau\, \overline{Y}_\theta\,. \tag{1.72}$$

In the flat case one poses $\gamma_1 = 0$, which is equivalent to drop the division by $1 + s$. So we get:

$$\lambda_0 = s\, \frac{\mathbf{U}_0(Z_1)}{\overline{L}} \sum_{\tau=1}^{T} \tau\, \overline{Y}_\theta\,. \tag{1.73}$$

The corresponding expressions for the LM approximation are as follows. For the discounted LM approximation (1.38):

$$K_0 = \frac{\mathbf{U}_0(D)}{1 + v_1\, s} \qquad \text{and} \qquad \lambda_0 = \frac{s}{1 + v_1\, s}\, \frac{\mathbf{U}_0(D)}{M_0} \sum_{\tau=1}^{T} v_\tau\, \widehat{M}_{\tau-1}\,. \tag{1.74}$$

For the φ-discounted LM approximation:

$$K_0 = \frac{\varphi\,\mathbf{U}_0(L)}{1 + v_1\, s} \qquad \text{and} \qquad \lambda_0 = \frac{s}{1 + v_1\, s}\,\frac{\mathbf{U}_0(L)}{\overline{L}} \sum_{\tau=1}^{T} v_\tau\, \widehat{M}_{\tau-1}\,. \tag{1.75}$$

For the undiscounted LM case:

$$K_0 = \frac{\mathbf{U}_0(L)}{1 + s} \qquad \text{and} \qquad \lambda_0 = \frac{s}{1 + s}\,\frac{\mathbf{U}_0(L)}{\overline{L}} \sum_{\tau=1}^{T} \tau\,\overline{Y}_\theta\,. \tag{1.76}$$

The LM approximation in the flat case gives:

$$K_0 = \mathbf{U}_0(L) \qquad \text{and} \qquad \lambda_0 = s\,\frac{\mathbf{U}_0(L)}{\overline{L}} \sum_{\tau=1}^{T} \tau\,\overline{Y}_\theta\,. \tag{1.77}$$

1.3.3 A tabular representation of different approaches and approximations

It could be useful to summarize in tabular form the alternative methods and approximations we considered for computing risk margins and risk capital. Table 1.1 reports in schematic form three basic approaches to the definition of the required reserve, where the risk margins are determined only by considering the p.d. of the "relevant random variable" (RRV), that is the variable chosen to express the insurer's obligations. We referred here to a required reserve computed as a quantile, but the representation is valid referring to reserve defined as any summary statistics of the RRV L or D.

In table 1.2 we summarize the four possible approximations in reserve risk capital computation under both the YEE approach and the LM approach. In this case we assume that the risk margin λ_0 in the required reserve, or at least the risk loading γ_1 on the first-year liability Y_1, have been already defined in a previous stage. Hence the "U-correction" term, that is the term correcting the unanticipated value $\mathbf{U}_0$ of the RRV, has been specified independently of the risk capital valuation.

In table 1.3 we represent the same approaches assuming that both the reserve risk capital and the risk margin are determined simultaneously, the risk margin λ_0 being defined as the cost κ_0 of the risk capital. In all of the methods considered, once the RRV has been chosen only the RAV $\mathbf{W}_0$ and the spread s have to be specified. The former provides the unanticipated value of the RRV; the latter (which expresses the market price of risk) determines the U-correction term (in this case a divisor) for deriving risk capital. The corresponding risk margin is obtained as the product of the s-, u- and μ- factor, as in the representation (1.65). The liability factor $\widehat{\mu}$ is determined by discounted or undiscounted expected liabilities depending on the approximation used.

	RRV	D
exact method	BE	$\mathbf{E}_0(D) = M_0$
(discounted)	RR	$R^* = \mathbf{Q}_0^{(\alpha)}(D)$
	RM	$\lambda_0 = \mathbf{Q}_0^{(\alpha)}(D) - M_0$
	RRV	$\varphi\, L$
φ-discount	BE	$\varphi\, \mathbf{E}_0(L) = \varphi\, \overline{L} = M_0$
approximation	RR	$R^* = \varphi\, \mathbf{Q}_0^{(\alpha)}(L)$
	RM	$\lambda_0 = \varphi\, \mathbf{Q}_0^{(\alpha)}(L) - M_0$
	RRV	L
undiscounted	BE	$\mathbf{E}_0(L) = \overline{L}$
approximation	RR	$R^* = \mathbf{Q}_0^{(\alpha)}(L)$
	RM	$\lambda_0 = \mathbf{Q}_0^{(\alpha)}(L) - \overline{L}$

RRV: relevant random variable
BE: best estimate
RR: required reserve
RM: risk margin

Table 1.1: Required reserve and risk margin definition: exact and approximated methods (R^* as α-quantile)

Method		**YEE approach**	**LM approach**
discounted	RRV	$v_1\, M_1^-$	D
	RAV	$v_1\, \mathbf{W}_0(M_1^-)$	$\mathbf{W}_0(D)$
	BE	$v_1\, \mathbf{E}_0(M_1^-) = M_0$	$\mathbf{E}_0(D) = M_0$
	U-corr.	$-v_1\, \gamma_1$	$-v_1\, \gamma_1$
	RC	$v_1\, \mathbf{U}_0(M_1^-) - v_1\, \gamma_1$	$\mathbf{U}_0(D) - v_1\, \gamma_1$
φ-discount	RRV	$\varphi\, Z_1$	$\varphi\, L$
	RAV	$\varphi\, \mathbf{W}_0(Z_1)$	$\varphi\, \mathbf{W}_0(L)$
	BE	$\varphi\, \mathbf{E}_0(Z_1) = M_0$	$\varphi\, \mathbf{E}_0(L) = \varphi\, \overline{L} = M_0$
	U-corr.	$-v_1\, \gamma_1$	$-v_1\, \gamma_1$
	RC	$\varphi\, \mathbf{U}_0(Z_1) - v_1\, \gamma_1$	$\varphi\, \mathbf{U}_0(L) - v_1\, \gamma_1$
undiscount	RRV	Z_1	L
	RAV	$\mathbf{W}_0(Z_1)$	$\mathbf{W}_0(L)$
	BE	$\mathbf{E}_0(Z_1) = \overline{L}$	$\mathbf{E}_0(L) = \overline{L}$
	U-corr.	$-\gamma_1$	$-\gamma_1$
	RC	$\mathbf{U}_0(Z_1) - \gamma_1$	$\mathbf{U}_0(L) - \gamma_1$
flat	RRV	Z_1	L
	RAV	$\mathbf{W}_0(Z_1)$	$\mathbf{W}_0(L)$
	BE	$\mathbf{E}_0(Z_1) = \overline{L}$	$\mathbf{E}_0(L) = \overline{L}$
	U-corr.	0	0
	RC	$\mathbf{U}_0(Z_1)$	$\mathbf{U}_0(L)$

Table 1.2: Risk capital valuation methods under YEE and LM approach: define RAV (risk loading γ_1 specified)

Method		**YEE approach**	**LM approach**
discounted	RRV	$v_1\, M_1^-$	D
	RAV	$v_1\, \mathbf{W}_0(M_1^-)$	$\mathbf{W}_0(D)$
	BE	$v_1\, \mathbf{E}_0(M_1^-) = M_0$	$\mathbf{E}_0(D) = M_0$
	U-corr.	$/(1 + v_1\, s)$	$/(1 + v_1\, s)$
	RC	$v_1\, \mathbf{U}_0(M_1^-)/(1 + v_1\, s)$	$\mathbf{U}_0(D)/(1 + v_1\, s)$
	s-factor	$\widehat{s} = s/(1 + v_1\, s)$	$\widehat{s} = s/(1 + v_1\, s)$
	μ-factor	$\widehat{\mu} = \sum_{\tau=1}^{T} v_\tau\, \widehat{M}_{\tau-1}$	$\widehat{\mu} = \sum_{\tau=1}^{T} v_\tau\, \widehat{M}_{\tau-1}$
	RM	$\kappa_0 = \widehat{s}\, \dfrac{\mathbf{U}_0(M_1^-)}{\mathbf{E}_0(M_1^-)}\, \widehat{\mu}$	$\kappa_0 = \widehat{s}\, \dfrac{\mathbf{U}_0(D)}{\mathbf{E}_0(D)}\, \widehat{\mu}$
φ-discount	RRV	$\varphi\, Z_1$	$\varphi\, L$
	RAV	$\varphi\, \mathbf{W}_0(Z_1)$	$\varphi\, \mathbf{W}_0(L)$
	BE	$\varphi\, \mathbf{E}_0(Z_1) = M_0$	$\varphi\, \mathbf{E}_0(L) = \varphi\, \overline{L} = M_0$
	U-corr.	$/(1 + v_1\, s)$	$/(1 + v_1\, s)$
	RC	$\varphi\, \mathbf{U}_0(Z_1)/(1 + v_1\, s)$	$\varphi\, \mathbf{U}_0(L)/(1 + v_1\, s)$
	s-factor	$\widehat{s} = s/(1 + v_1\, s)$	$\widehat{s} = s/(1 + v_1\, s)$
	μ-factor	$\widehat{\mu} = \sum_{\tau=1}^{T} v_\tau\, \widehat{M}_{\tau-1}$	$\widehat{\mu} = \sum_{\tau=1}^{T} v_\tau\, \widehat{M}_{\tau-1}$
	RM	$\kappa_0 = \widehat{s}\, \dfrac{\mathbf{U}_0(Z_1)}{\mathbf{E}_0(Z_1)}\, \widehat{\mu}$	$\kappa_0 = \widehat{s}\, \dfrac{\mathbf{U}_0(L)}{\mathbf{E}_0(L)}\, \widehat{\mu}$
undiscount	RRV	Z_1	L
	RAV	$\mathbf{W}_0(Z_1)$	$\mathbf{W}_0(L)$
	BE	$\mathbf{E}_0(Z_1) = \overline{L}$	$\mathbf{E}_0(L) = \overline{L}$
	U-corr.	$/(1 + s)$	$/(1 + s)$
	RC	$\mathbf{U}_0(Z_1)/(1 + s)$	$\mathbf{U}_0(L)/(1 + s)$
	s-factor	$\widehat{s} = s/(1 + s)$	$\widehat{s} = s/(1 + s)$
	μ-factor	$\widehat{\mu} = \sum_{\tau=1}^{T} \sum_{\theta=\tau}^{T} \overline{Y}_\theta$	$\widehat{\mu} = \sum_{\tau=1}^{T} \sum_{\theta=\tau}^{T} \overline{Y}_\theta$
	RM	$\kappa_0 = \widehat{s}\, \dfrac{\mathbf{U}_0(Z_1)}{\mathbf{E}_0(Z_1)}\, \widehat{\mu}$	$\kappa_0 = \widehat{s}\, \dfrac{\mathbf{U}_0(L)}{\mathbf{E}_0(L)}\, \widehat{\mu}$
flat	RRV	Z_1	L
	RAV	$\mathbf{W}_0(Z_1)$	$\mathbf{W}_0(L)$
	BE	$\mathbf{E}_0(Z_1) = \overline{L}$	$\mathbf{E}_0(L) = \overline{L}$
	U-corr.	$/1$	$/1$
	RC	$\mathbf{U}_0(Z_1)$	$\mathbf{U}_0(L)$
	s-factor	$\widehat{s} = s$	$\widehat{s} = s$
	μ-factor	$\widehat{\mu} = \sum_{\tau=1}^{T} \sum_{\theta=\tau}^{T} \overline{Y}_\theta$	$\widehat{\mu} = \sum_{\tau=1}^{T} \sum_{\theta=\tau}^{T} \overline{Y}_\theta$
	RM	$\kappa_0 = s\, \dfrac{\mathbf{U}_0(Z_1)}{\mathbf{E}_0(Z_1)}\, \widehat{\mu}$	$\kappa_0 = s\, \dfrac{\mathbf{U}_0(L)}{\mathbf{E}_0(L)}\, \widehat{\mu}$

Table 1.3: Risk margin and risk capital valuation methods under YEE and LM approach (define RAV and specify s)

Chapter 2

Modelling P&C liabilities

2.1 Run-off triangles

In P&C insurance the OLL generated by a given portfolio of policies can be formally represented referring to the triangle of claim payments (paid losses) made up to the current date $t = 0$. We assume that observations of past payments are referred to claims originated in the time period $[-n, 0]$, with n integer. Therefore data from the past n *accident years* (AY) are available. It is convenient to assume the current date as the end of a calendar year; therefore any accident year coincides with an accounting year and the valuation at time $t = 0$ can be considered as being made at the end of the n-th of the just observed accounting years. In our application $t = 0$ denotes the end of 2004. For any AY data are assumed to be organized by *development year* (DY), which is the delay between the accident date and the payment date.

For notational convenience the indication of the valuation date will be dropped if unnecessary and the suffixes will be used to indicate accident years and development years.

2.1.1 Paid losses

Incremental paid losses

Let us denote by:

$$C_{i,j}\,, \quad i = 1, \ldots, n\,, \quad j = 1, \ldots, J\,,$$

the *incremental paid losses*, i.e. the portfolio payments relative to claims originated in the AY i and made in the DY j. In general $J \geq n$. If $J > n$, the quantity:

$$T_\infty := \sum_{j=n+1}^{J} C_{1,j}\,,$$

i.e. the sum of the incremental losses originated in the oldest AY that are not yet observed, represents the *tail* of the outstanding losses.

Let us assume for the moment that $J = n$. Hence $T_\infty = 0$ and the effects of claims originated in the first AY are completely observable at time zero; no provision is needed on that date for this generation of liabilities. The paid losses $C_{i,j}$ can now be arranged into an $n \times n$ matrix; the elements on the diagonal, that is the elements:

$$C_{i,j} \quad \text{such that} \quad i + j = n + 1\,,$$

are the payments made in the most recent calendar year $[-1,0)$. The payments such that $i+j \le n+1$ are made before the current date $t=0$; they form the "past triangle", which is the set of the observed data. The elements of the "future triangle", i.e. the $C_{i,j}$ such that $i+j > n+1$, are random variables at time $t=0$.

Cumulative paid losses

The *cumulative paid losses* are defined as:

$$S_{i,j} = \sum_{k=1}^{j} C_{i,k}\,, \qquad i,j = 1,\dots,n\,.$$

It is useful to introduce a notation denoting, for each AY, the DY most recently observed. For any $k = 1,\dots,n$, define the *diagonal index* as:

$$d_k = n - k + 1\,;$$

the index d_k is the column index of the diagonal element on the row k (or, equivalently, the row index of the diagonal element on the column k). Thus the total payments made up to time $t=0$ for claims of the AY i are given by S_{i,d_i}.

The *total payments* up to time zero are given by:

$$S = \sum_{i=1}^{n} S_{i,d_i}\,.$$

2.1.2 Ultimate losses and outstanding losses

Ultimate losses

The *ultimate losses* are the cumulative paid losses on the latest DY; we have:

$$U_i = S_{i,n}\,,\ i = 1,\dots,n\,, \qquad U = \sum_{i=1}^{n} U_i\,,$$

where U is the total ultimate loss.

Outstanding Loss Liabilities

The *outstanding loss liabilities* (OLL) are the part of the ultimate loss that are not yet paid at time $t=0$, i.e.:

$$L_i = U_i - S_{i,d_i}\,,\ i = 1,\dots,n\,, \qquad L = \sum_{i=1}^{n} L_i\,.$$

Of course by the assumption $J = n$ one has $U_1 = S_{1,n}$ and $L_1 = 0$.

Ultimate loss reserve

If not otherwise stated all the probability assessments are assumed to be made based on the information available at time $t = 0$. When conditioning on information at time t has to be explicitly specified the suffix t will be used.

We shall denote by:

$$\overline{L}_i := \mathbf{E}(L_i)\,, \quad i = 1, \ldots, n\,,$$

the expectation (at time $t = 0$) of the OLL originated in AY i. The expected value:

$$\overline{L} = \sum_{i=1}^{n} \overline{Y}_i\,,$$

is interpreted as the best estimate of the total future liabilities and is usually assumed as the *ultimate loss reserve* of the outstanding portfolio. This definition does not include any explicit risk margin, apparently assuming that the valuation on an undiscounted basis implicitly provides an adequate risk loading. In order to obtain the more general definition of a required reserve R^* both discounting and appropriate explicit risk margins should be included.

2.1.3 Payments in future years

The decomposition by AY of total paid losses is convenient for applying efficient forecasting methods of future liabilities. However it is not well suited for fair valuation, where the maturity structure of the OLL must be specified.

Assuming that the payments are made at the end of each year, the amount $C_{i,j}$ will fall due after $\tau = j - d_i$ years from the date $t = 0$. Hence the sum:

$$Y_\tau := \sum_{i=\tau+1}^{n} C_{i,d_i+\tau}\,, \qquad \tau = 1, \ldots, n-1\,,$$

represents the total claims payment to be made at time τ (i.e. at the end of the accounting year τ). Of course one has:

$$\overline{L} = \sum_{\tau=1}^{n-1} \overline{L}_\tau\,,$$

that is the ultimate loss reserve can also be obtained summing by diagonal the expected value of all future payments.

The diagonal payments Y_τ describe the correct maturity structure of the OLL and are suitable for providing required reserve definitions on a discounted basis, along the principles introduced in section 1.2. Provided that an appropriate set $\{\gamma_\tau;\, \tau = 1, \ldots, n-1\}$ of market risk loadings for the liabilities Y_τ is defined, the fair value of the OLL is given by:

$$V = \sum_{\tau=1}^{n-1} v_\tau\, (\overline{Y}_\tau + \gamma_\tau)\,.$$

Under the FV assumption this expression provides the definition of the required reserve R^*.

2.2 Reserve process and risk capital

Since all claim payments are assumed to be made at the end of each year, at the date $t = 1$ and immediately before (say at the end of the next year) the insurer's obligations are given by the claim losses:

$$Y_1 = \sum_{i=2}^{n} C_{i,d_i+1} , \tag{2.1}$$

generated by the outstanding portfolio in the accounting year $[0, 1)$, plus the reserve amount R_1^* representing the provisions for the remaining OLL. Of course, under the FV assumption $R_\tau^* = V_\tau$ ($\tau = 0, 1, \ldots, n-1$) and the year-end obligations are $V_1^- := Y_1 + V_1$.

In the flat case (neither discounting nor risk margins) the reserve coincides with the ultimate loss reserve, hence:

$$V_0 = \overline{L} \quad \text{and} \quad V_1 = \sum_{\tau=2}^{n-1} \mathbf{E}_1(Y_\tau) ,$$

and we adopted the notation:

$$Z_1 := V_1^- = Y_1 + \sum_{\tau=2}^{n-1} \mathbf{E}_1(Y_\tau) = \sum_{\tau=1}^{n-1} \mathbf{E}_1(Y_\tau) .$$

In this case the reserve can be decomposed also by AY; that is we can write:

$$Z_1 = \sum_{i=2}^{n} C_{i,d_i+1} + \sum_{\tau=2}^{n-1} \sum_{i=\tau+1}^{n} \mathbf{E}_1\big(C_{i,d_i+\tau}\big) = \sum_{i=2}^{n} \sum_{j=d_i+1}^{n} \mathbf{E}_1\big(C_{i,j}\big) .$$

This can also be written:

$$Z_1 = \sum_{i=2}^{n} \big[\mathbf{E}_1\big(U_i\big) - S_{i,d_i}\big] = \mathbf{E}_1\big(U\big) - S ; \tag{2.2}$$

hence in the flat case the r.v. expressing the year-end obligations can be expressed as the expectation at $t = 1$ of the total ultimate losses minus the cumulative paid losses at time $t = 0$. Obviously $\mathbf{E}_0(Z_1) = \mathbf{E}_0\big(U\big) - S = \overline{L}$.

As concerning measures of reserve risk capital all the definitions discussed in section 1.2 can be adopted and the corresponding formulas can be applied, provided that the terminal payment date T is specified as $n - 1$, which is the last accounting year in the future triangle. Of course the traditional decomposition by AY of the relevant quantities will be viable only if the linearity property holds, that is only in the flat case.

2.3 Run-off techniques for P&C liabilities

Run-off techniques for P&C claims reserving are forecasting methods based on the assumptions that there is a consistent pattern in the past claim experience. These methods can be qualified as deterministic. They produce a point estimate of the ultimate losses L_i projecting into the future the development of claims payments observed in the past triangle, without specifying any underlying probabilistic assumption.

2.3.1 The chain-ladder algorithm

The development factors

The chain-ladder method is probably the most popular run-off technique. For $i = 2, \ldots, n$, it provides an estimate of the future cumulative paid losses by the projection rules:

$$\begin{cases} \widehat{S}_{i,d_i+1} &= S_{i,d_i}\,\lambda_{d_i}\,, \\ \widehat{S}_{i,j+1} &= \widehat{S}_{i,j}\,\lambda_j\,, \quad j = d_i+1, d_i+2, \ldots, n-1\,, \end{cases} \tag{2.3}$$

where the factors λ_j are the *individual development factors* (or *individual link ratios*) defined as:

$$\lambda_j = \frac{\sum_{i=1}^{d_j-1} S_{i,j+1}}{\sum_{i=1}^{d_j-1} S_{i,j}}\,, \quad j = 1, \ldots, n-1\,. \tag{2.4}$$

The *cumulative development factors* Λ_j are the development factors from the DY j on, that is:

$$\Lambda_j = \prod_{k=j}^{n-1} \lambda_k\,, \qquad j = 1, \ldots, n-1\,. \tag{2.5}$$

By this definition the cumulative paid losses at the latest DY can be expressed starting from the current cumulative payments:

$$\widehat{S}_{i,n} = S_{i,d_i}\,\Lambda_{d_i}\,, \qquad i = 1, 2, \ldots, n\,.$$

Hence the estimated ultimate losses are given by:

$$\widehat{U}_i = \widehat{S}_{i,n}\,, \quad i = 1, 2, \ldots, n\,, \qquad \widehat{U} = S_{1,n} + \sum_{i=1}^{n} \widehat{U}_i\,,$$

and the estimated OLL are:

$$\widehat{L}_i = \widehat{U}_i - S_{i,d_i}\,, \quad i = 1, 2, \ldots, n\,, \qquad \widehat{L} = \sum_{i=1}^{n} \widehat{L}_i\,.$$

Obviously $\widehat{L}_1 = 0$ since under the assumption $J = n$ one has $\widehat{U}_1 = \widehat{S}_{1,n} = S_{1,n}$.

Remark. Definitions (2.4) and (2.5) can be obviously extended up to DY n posing $\Lambda_n := \lambda_n := 1$. ■

Including the tails

If the assumption $J = n$ is relaxed the estimated ultimate losses must be incremented to take into account the positive tail T_∞. A viable approximation of the estimation problem can be obtained substituting the incremental paid loss $C_{1,n}$ by:

$$C_{1,n}^{+} := C_{1,n} + \Delta_1\,,$$

where Δ_1 is an exogenous estimate of T_∞, and assuming again $J = n$. Hence one is led to a square-matrix problem and all the previous definitions apply under suitable adjustments. In particular, since the cumulative payment $S_{1,n}$ is substituted by $S^+_{1,n} := S_{1,n} + \Delta_1$, the latest cumulative development factor to be considered is given by:

$$\Lambda_n := \lambda_n := \frac{S^+_{1,n}}{S_{1,n}} = 1 + \frac{\Delta_1}{S_{1,n}} \, .$$

All the estimated ultimate losses $\widehat{U}_i$ are then modified by the factor Λ_n; one has:

$$\widehat{U}^+_i = \widehat{U}_i \, \Lambda_n \, , \quad i = 1, \ldots, n \, , \qquad\qquad \widehat{U}^+ = \widehat{U} \, \Lambda_n \, .$$

Of course the OLL are derived referring to the claims actually paid, that is:

$$\widehat{L}^+_i = \widehat{U}^+_i - S_{i,d_i} \, , \quad i = 1, \ldots, n \, , \qquad\qquad \widehat{L}^+ = \sum_{i=1}^{n} \widehat{L}^+_i \, .$$

Similarly, the future-years payoffs are given by:

$$\begin{cases} \widehat{Y}^+_1 = \Delta_1 + \widehat{C}^+_{2,n} + \displaystyle\sum_{i=3}^{n} \widehat{C}_{i,d_i+1} \, , \\ \widehat{Y}^+_\tau = \widehat{C}^+_{\tau+1,n} + \displaystyle\sum_{i=\tau+2}^{n} \widehat{C}_{i,d_i+\tau} \, , \qquad \tau = 2, \ldots, n-1 \, , \end{cases}$$

where $\widehat{C}^+_{i,j}$ are the incremental payments estimated applying the chain-ladder algorithm to the past triangle where $S_{1,n}$ is substituted by $S^+_{1,n}$.

2.4 Stochastic reserving models

Traditional run-off techniques are not able to provide risk margin and risk capital measures since they produce only point estimates, without any assessment of the OLL variability. The natural way to obtaining risk measures is to derive the full probability distribution of future claim payments, or at least to add the estimate of a second order moment to the OLL point estimate. In recent years a number of stochastic models has been proposed in general insurance, extending the traditional deterministic techniques and providing a probabilistic representation of the OLL. In this analysis we used two of the most popular stochastic models, suitable to be applied to our triangles of paid losses data. Both of these models can be considered as a stochastic extension of the chain-ladder method, providing expected values identical to the projected values given by the classical deterministic approach.

Before presenting the models some general considerations are in order concerning measures of variability in statistical modelling.

2.4.1 Variability of OLL under stochastic models

In a stochastic framework claim reserving is a predictive process, where forecasts of future claims are derived based on the observed data. It is important to observe that in this predictive process the full variability of the OLL includes both the inherent variability in the data being forecast and the uncertainty in parameters estimation. Let us refer, for example, to the random variable L representing the overall OLL. Estimating a suitable stochastic model for L on data on the past triangle a "predicted value" $\widehat{L}$ will be obtained representing an estimate of $\mathbf{E}(L)$. Under classical statistical methods the "true value" of $\mathbf{E}(L)$ is unknown and given that the data are considered a random observation sample, the estimator $\widehat{L}$ is a random variable also. Thus the variability of $\widehat{L}$ includes both the variability of the r.v. L and the variability of the estimate. We are interested in the total variability, i.e. in the *prediction variance* (or (*squared*) *prediction error*) of $\widehat{L}$, which is defined as:

$$\mathbf{Pvar}(\widehat{L}) := \mathbf{E}\Big[\big(L - \widehat{L}\big)^2\Big] .$$

Typically an approximation of the prediction variance is obtained assuming unbiasedness (i.e. $\mathbf{E}(L) = \mathbf{E}(\widehat{L})$) and independence between past observations and future observations (e.g. see [21]). The following decomposition holds:

$$\mathbf{Pvar}(\widehat{L}) \approx \mathbf{Var}(L) + \mathbf{Evar}(\widehat{L}) , \tag{2.6}$$

where:

$$\mathbf{Evar}(\widehat{L}) := \mathbf{Var}(\widehat{L}) = \mathbf{E}\Big[\big(\widehat{L}_i - \mathbf{E}(\widehat{L})\big)^2\Big] , \tag{2.7}$$

is the *estimation variance* (or (*squared*) *estimation error*). In this decomposition the variance of L:

$$\mathbf{Var}(L) := \mathbf{E}\Big[\big(L - \mathbf{E}(L)\big)^2\Big] , \tag{2.8}$$

is usually referred to as the *process variance* (or (*squared*) *process error*). Of course the prediction error and the estimation error are defined as the corresponding standard deviations:

$$\mathbf{Pstd}(\widehat{L}) := \sqrt{\mathbf{Pvar}(\widehat{L})} , \qquad \mathbf{Estd}(\widehat{L}) := \sqrt{\mathbf{Evar}(\widehat{L})} . \tag{2.9}$$

2.5 The Distribution Free stochastic Chain-Ladder (Mack's model)

The "distribution free stochastic chain-ladder" (DFCL) suggested by Mack in 1993 [30] assumes that the payments of different AY are independent and the conditional mean and variance of the cumulative payments $S_{i,j}$ are, respectively:

$$\mathbf{E}\big(S_{i,j}|S_{i,j-1}\big) = \lambda_{j-1}\, S_{i,j-1} , \qquad \mathbf{Var}\big(S_{i,j}|S_{i,j-1}\big) = \omega_{j-1}^2\, S_{i,j-1} , \tag{2.10}$$

where λ_j and ω_j ($j = 1, 2, \ldots, n-1$) are unknown parameters.

2.5.1 The parameter estimators

The parameters λ_{j-1} are the individual development factors of the model and must be estimated, together with the standard deviation factors ω_{j-1}, from observed data. As shown by Mack, the estimators $\widehat{\lambda}_{j-1}$ of the development factors are the analogous of the chain-ladder estimators:

$$\widehat{\lambda}_{j-1} = \frac{\sum_{i=1}^{d_j} S_{i,j}}{\sum_{i=1}^{d_j} S_{i,j-1}} , \quad j = 1, \ldots, n . \tag{2.11}$$

Moreover the appropriate estimators of ω^2_{j-1} are give by:

$$\widehat{\omega}^2_{j-1} := \frac{1}{n-j} \sum_{i=1}^{d_j} S_{i,j-1} \left(\frac{S_{i,j}}{S_{i,j-1}} - \widehat{\lambda}_{j-1} \right)^2 , \quad j = 1, 2, \ldots, n-1 ; \tag{2.12}$$

for data on a triangular array the estimator of the latest parameter ω^2_{n-1} can be fixed as:

$$\widehat{\omega}^2_{n-1} := \min \left(\frac{\widehat{\omega}^4_{n-2}}{\widehat{\omega}^2_{n-3}}, \widehat{\omega}^2_{n-3} \right) .$$

2.5.2 Prediction errors of the OLL

OLL of a single accident year

The estimated values of λ_{j-1} provide the corresponding estimate $\widehat{L}_i$ of the OLL. For the ultimate losses one has:

$$\widehat{U}_i = \widehat{\mathbf{E}}_t(U_i) = S_{i,d_i} \, \widehat{\Lambda}_{d_i} ;$$

hence for the OLL one obtains:

$$\widehat{L}_i = \widehat{U}_i - S_{i,d_i} = S_{i,d_i} \left(\widehat{\Lambda}_{d_i} - 1 \right) .$$

Since the full predictive distribution is not provided by the model we consider the variability expressed by the prediction variance of $\widehat{L}_i$:

$$\mathbf{Pvar}(\widehat{L}_i) := \mathbf{E}\Big[\big(L_i - \widehat{L}_i \big)^2 \Big] .$$

Obviously the prediction variance of $\widehat{L}_i$ is equal to the prediction variance of $\widehat{U}_i$, since the cumulative payments S_{i,d_i} are known at time t.

As usual the prediction variance can be approximated as the sum of the estimation variance and the process variance:

$$\mathbf{Pvar}(\widehat{L}_i) \approx \mathbf{Evar}(\widehat{L}_i) + \mathbf{Var}(L_i) .$$

Mack produces explicit expressions of both the components of the prediction variance $\mathbf{Pvar}(\widehat{L}_i)$. A correction for the expression of the estimation variance is given in [3].

Overall OLL

A crucial point is the derivation of the prediction error of the overall liabilities $\widehat{L}$ which is usually of interest in the actuarial practice. The r.v.s $S_{i,j}$ corresponding to different AY are independent by assumption. However the prediction variance of $\widehat{L}$ cannot be obtained as the sum of the single prediction variances $\mathbf{Pvar}(\widehat{L}_i)$ of the single AY liabilities. A correct expression for $\mathbf{Pvar}(\widehat{L})$ can be found in [3]. In the same paper and in [34] the appropriate expressions can also be found of the prediction errors of the estimators $\widehat{Y}_\tau$ of future years payments, which are of interest in our applications.

Quantiles from lognormal assumption

Since the Mack model is distribution free, additional assumptions are needed if the computation of summary statistics of the OLL different from the first two moments are required. In order to derive quantiles and expected shortfalls of the OLL we assume a lognormal distribution with mean and variance equal to $\widehat{L}$ and $\mathbf{Pvar}(\widehat{L})$, respectively.

2.6 The Over-Dispersed Poisson Model

In the "over-dispersed Poisson" (ODP) model [35] the incremental paid losses $C_{i,j}$ are independent over-dispersed Poisson r.v.s, with mean and variance given by:

$$\mathbf{E}(C_{i,j}) = m_{i,j}\,, \qquad \mathbf{Var}(C_{i,j}) = \phi\, m_{i,j}\,, \qquad i,j = 1,\ldots,n\,,$$

where:

$$m_{i,j} := x_i\, y_j\,,$$

with x_i, y_j, $\phi > 0$, and:

$$\sum_{k=1}^{n} y_k = 1\,.$$

Since:

$$\mathbf{E}(U_i) = \sum_{j=1}^{n} \mathbf{E}(C_{i,j}) = \sum_{j=1}^{n} x_i\, y_j = x_i\,, \tag{2.13}$$

one can see that the "row parameter" x_i represents the expected ultimate loss of the AY i. Moreover the "column parameter":

$$y_j \equiv \frac{\mathbf{E}(C_{i,j})}{\mathbf{E}(U_i)}\,,$$

represents the proportion of the expected ultimate loss to emerge in the DY j for any AY.

The ODP model and the chain-ladder algorithm

In [37] it is shown that, under the ODP assumption (and some suitable additional conditions), the maximum likelihood estimators $\widehat{\lambda}_j$ of the individual development factors can be obtained by the maximum likelihood estimates $\widehat{y}_j$ of the column parameters:

$$\widehat{\lambda}_j = \frac{\sum_{k=1}^{j+1} \widehat{y}_k}{\sum_{k=1}^{j} \widehat{y}_k}\,, \qquad j = 1,\ldots,n-1\,.$$

This relations provide the same development factors given by (2.4). Thus the chain-ladder technique can be viewed as a method for deriving maximum likelihood estimates which are consistent with the ODP model. So if one considers the chain-ladder as an acceptable projection method, then one can consistently assume the ODP model to describe the stochastic nature of the paid losses[1]. As for the DFCL, the chain-ladder projections can be considered as the expectations provided by the stochastic model.

[1] Some controversies have been roused discussing which stochastic model exactly underlies the chain-ladder technique. See [37], [31], [38] and [21] for a discussion of this point.

2.6.1 The ODP model as a generalized linear model

For estimation purposes the ODP model can be reparameterised posing:

$$\begin{aligned} \log(m_{i,j}) &:= \eta_{i,j}\,, \\ \eta_{i,j} &:= c + \alpha_i + \beta_j\,, \qquad \alpha_1 = \beta_1 = 0\,. \end{aligned} \tag{2.14}$$

Thus a generalized linear model (GLM) is assumed where the response is modelled by a logarithmic link function and the variance is proportional to the mean through the scale parameter ϕ. Such a model can be easily estimated using standard software, which provides both the estimators $\widehat{\eta}_{i,j}$ and the estimation variances $\mathbf{Var}(\widehat{\eta}_{i,j})$. The estimates of incremental paid losses are:

$$\widehat{C}_{i,j} = \widehat{m}_{i,j} := e^{\widehat{\eta}_{i,j}}\,. \tag{2.15}$$

It can be shown (see [21] for example) that the corresponding prediction variance can be approximated as:

$$\mathbf{Pvar}(\widehat{C}_{i,j}) \approx \phi\,\widehat{C}_{i,j} + \widehat{m}^2_{i,j}\,\mathbf{Var}(\widehat{\eta}_{i,j})\,. \tag{2.16}$$

Prediction errors of the OLL

For determining the prediction variance of the liabilities L_i from each AY and of the overall liabilities L the appropriate covariance terms $\mathbf{Cov}(\widehat{C}_{i_1,j_1}, \widehat{C}_{i_2,j_2})$ must be taken into account. Proceeding as with the variance terms, the covariances can be approximated as:

$$\mathbf{Cov}(\widehat{C}_{i_1,j_1}, \widehat{C}_{i_2,j_2}) \approx \widehat{m}_{i_1,j_1}\,\widehat{m}_{i_2,j_2}\,\mathbf{Cov}(\widehat{\eta}_{i_1,j_1}, \widehat{\eta}_{i_2,j_2})\,. \tag{2.17}$$

Thus they can be directly computed if the covariance matrix of the parameter estimates $\widehat{\eta}_{i,j}$ is available from the software package used to estimate the GLM model.

2.6.2 Deriving full predictive distributions by simulation

Obviously the first two moments of L give only limited information on the probability distribution of the OLL, which is required if other summary statistics, such as measures of skewness or extreme percentiles are also of interest. In order to derive the full predictive distribution of the possible OLL outcomes under the ODP model we used a simulation procedure. In this approach the variability of the estimates is simulated by bootstrapping the residuals on the fitted values provided by the chain-ladder technique. The variability given by the process variance is generated by adding to the projected incremental paid losses a random error sampled from an (approximated) ODP distribution. This approach was first used by [20] and [19].

It is worthwhile to observe that the simulation procedure produces an empirical predictive distribution for each incremental payment $C_{i,j}$ in the future triangle. Thus the predictive distribution of the liabilities L_i from each AY, of the payoffs Y_τ in each future accounting year and the overall liabilities $L = \sum_{i=2}^{n} L_i = \sum_{\tau=1}^{n-1} Y_\tau$ are immediately obtained aggregating the simulated incremental paid losses by row, by diagonal and over the entire future triangle, respectively. The distribution of the discounted liabilities $v_\tau\,Y_\tau$ and of the total discounted liabilities $D = \sum_{\tau=1}^{n-1} v_\tau\,Y_\tau$ can also be obtained by the same method. With this procedure the diversification effects at different aggregation levels, i.e. across individual paid losses, across accident years or across accounting years, are naturally accounted for.

Measuring risk capital by simulation

The bootstrapped simulation procedure also allows to derive reliable measures of risk capital. As shown in section 1.2.6, in order to calculate the current year risk capital K_0 under the YEE approach the distribution of the year-end insurer's obligations

$$M_1^- = Y_1 + M_1 = Y_1 + \frac{1}{v_1} \sum_{\tau=2}^{n-1} v_\tau \, \mathbf{E}_1(Y_\tau)\,,$$

is required. Since the chain-ladder provides consistent estimates of the expected future payments under the ODP model, the year-end expectations $\mathbf{E}_1(Y_\tau)$ are readily derived in each iteration enlarging the past triangle with the simulated paid losses in the next-year diagonal and then applying the chain-ladder algorithm to this updated triangle. The collection of the estimated expectations and of there discounted values produced in each iteration provides the empirical distribution of the year-end obligations, in both the undiscounted and the discounted case.

This approach to the risk capital computation under the YEE method was introduced in [13] which is also referred to for technical details. Of course risk capital measures defined with the LM approach, also described in section 1.2.6, are directly derived by the empirical distribution of the discounted and undiscounted liabilities Y_τ.

Part II

Analysis of market data

Chapter 3

The data

3.1 Original data

The analysis considered data on claim experience on Motor Third Party Liability (MTPL) from 55 Italian insurance companies, referring to accounting years from 1995 to 2004 (the "current calendar year"). In terms of statutory reserve these companies represented about the 98% of the total MTPL Italian market as of 2004. The main informations used were data on number of paid claims, claim payments (paid losses) and statutory reserves. In each accounting year ("calendar year", CY) data were organized by year of occurrence ("accident year", AY). For accounting years 1995 to 1999 only data for accident years starting from 1995 were available. From accounting year 2000 on, reported data included claims occurred in the previous 11 years; hence for CY $n = 2000, \ldots, 2004$ data from AY $n - 11$ were available. Defining the "development year" (DY) as the difference DY := CY − AY + 1, data for CY = $1995, \ldots, 1999$ contained figures for DY $1, \ldots, 5$, respectively; for CY $\geq$ 2000 data referred to DY $\leq$ 12 were available.

If data are organized under the usual array AY/DY (using AY as the row index and DY as the column index), a complete triangular array can be obtained only starting from AY 1995. The triangle from AY 1989 is incomplete, since data for AY 1989 to 1994 are not available for the early DY.

Given that the most popular run-off techniques (as the chain-ladder method) are well-suited for the application to complete triangular (or trapezoidal) data arrays, in a first step the analysis has been performed using only data from AY 1995. Cutting data at AY = 1995 implies that only observations for up to 10 development years are considered. The information on accident years prior to 1995 suggests however that the typical run-off period is longer than 10 years and a tail must then be added to the cut data; as tail value the level of statutory reserve fixed by each company for claims originated in 1995 has been chosen.

3.2 Reduction of the data sample

In a first application of the stochastic reserving models only triangles of paid losses have been used. Not all the available data were well suited to be analysed by chain-ladder type methods. A typical problem is that for new companies the time series of data can be too much short, and the payments are typically fast growing. Moreover in some cases data can be incomplete or contain evident errors. To enable a comparison of results across different companies, we considered only complete triangles containing reliable paid losses data from the latest 10 accident years. Hence 15 companies have been excluded from the analysis. In term of statutory reserve these companies represented only the 4.4% of the total reserve of the complete sample of 55 companies. The statutory reserve of the 40 companies considered in the analysis totalized at about the 93% of the overall MTPL market reserve as of the end of 2004.

In this report data on statutory reserve have been disclosed on an aggregate basis, grouping the companies of the selected sample in four dimensional classes determined by the amount of R^s. The classification is specified in table 3.1, where figures are expressed in million Euros.

Class	Statutory reserve	n. co.	Total reserve
1	$1,000 \leq R^s$	9	15,114.43
2	$250 \leq R^s < 1,000$	15	7,257.39
3	$50 \leq R^s < 250$	13	1,541.70
4	$0 \leq R^s < 50$	3	128.45
total		40	24,041.97

Table 3.1: Classification of companies of the selected sample by statutory reserve

Chapter 4

Applying the stochastic models to historical paid losses triangles

In a first step the ODP and the DFCL model have been applied to the ten-year triangles of paid losses of the companies in the selected sample. The ODP model have been applied with the bootstrap method with 10000 simulations, producing the full predictive distribution of each of the payed losses $C_{i,j}$ in the future triangles. By aggregation the distributions have been derived both of the OLL L_i by accident year and of the future payments Y_τ by accounting year, as well as of the total OLL $L = \sum_{i=1}^{n} L_i = \sum_{\tau=1}^{n-1} Y_\tau$. The predicted values and the prediction squared errors (as the sum of the process errors and the estimation errors) for the same quantities have been computed under the Mack's model using closed form expressions. The full distribution of L has been derived making the additional assumption that the overall OLL are lognormally distributed.

To simplify notations we shall usually omit the symbol "$\widehat{\ }$" which denotes the predicted value of the random variables. We emphasize however that all the probability distribution considered in the sequel both under the ODP and the DFCL model, have to be considered of predictive type, including both process uncertainty and estimation uncertainty.

As stochastic extensions of the deterministic chain-ladder, the ODP and the Mack's model should provide identical values of the expected OLL $\overline{L}$. However, since the ODP model is applied with a Monte Carlo procedure all figures produced by this model will be affected by a random error (Monte Carlo error). In particular the sample mean $\widetilde{L}$ computed on the overall distribution of the simulated OLL will be slightly different from the theoretical chain-ladder value $\overline{L}$. In the selected sample the average of the percentage difference $(\widetilde{L} - \overline{L})/\overline{L}$ over the 10000 simulations resulted of about 0.13%, with a minimum value of -0.99% and a maximum of 0.99%. For a better comparison between results obtained with the two models all the OLL figures produced by simulation under the ODP model (both discounted and undiscounted) have been adjusted for the Monte Carlo error subtracting the difference $(\widetilde{L} - \overline{L})$. When applied to $\widetilde{L}$ this adjustment will obviously provide a sample mean of L exactly equal to the chain-ladder mean $\overline{L}$, by construction.

In the valuations provided by the two models the theoretical value $\overline{L}$ has ben often used as the reference level, the monetary amounts concerning any individual company being usually expressed in percent of the company-specific value of $\overline{L}$.

4.1 First results for undiscounted liabilities

4.1.1 Required reserve as undiscounted mean and undiscounted quantile

Some preliminary results by the ODP model are provided in table 4.1. For each selected company we reported:

- the class of statutory reserve;
- the coefficient of variation of L (in percent):

$$\mathbf{Cv}(L) := \frac{\mathbf{Pstd}(L)}{\overline{L}}\,;$$

- the 75-th, 90-th and 95-th quantile, $\mathbf{Q}^{(75)}(L)$, $\mathbf{Q}^{(90)}(L)$ and $\mathbf{Q}^{(95)}(L)$, of L (in percent of $\overline{L}$).

The companies in the table are sorted by increasing values of $\mathbf{Cv}(L)$. The corresponding order number has ben adopted as company-specific code number; it is denoted by "CodCv" and will be maintained during all the analysis.

The analogous quantities have been derived by the DFCL model and are reported in table 4.2. Also in this case the companies are sorted by CodCv code, hence by increasing value of the ODP variability; the comparison between the corresponding values of $\mathbf{Cv}(L)$ in the two tables shows that the relation between the coefficients of variation provided by the ODP and the DFCL model in the sample is not monotonic.

4.1.2 Risk margins from quantiles

Referring to the undiscounted OLL, for any definition of the required reserve R^* a risk margin δ is obtained as the difference:

$$\delta := R^* - \overline{L}\,;$$

therefore definitions of required reserve that are more conservative than the sample mean imply a positive risk margin. In figures 4.1 and 4.2 the risk margins provided by the two models are illustrated assuming the 75-th, 90-th and 95-th quantile of L as the required reserve. The companies are sorted by CodCv (hence by increasing values of the ODP coefficient of variation) and the figures are expressed as a percentage of the chain-ladder sample mean:

$$\mathrm{RM}^{(\alpha)} := \frac{\mathbf{Q}^{(\alpha)}(L) - \overline{L}}{\overline{L}}\,,$$

with $\alpha = 75,\ 90,\ 95\%$. The RM values at the three different confidence levels are represented with dots joined by solid line, dashed line and dotted line, respectively and are reported on the left vertical axis. A conventional value of statutory reserve for each dimensional class has been defined, attributing the symbolic value of $1500, 500, 100$ and 50 to class $1, 2, 3$ and 4, respectively. These conventional values are also reported on the right vertical axis in the figures and represented by vertical bars.

Of course the risk margins under the DFCL model are directly influenced by the lognormality assumption.

CodCv	Class	$\mathbf{Cv}(L)$ (%)	$\mathbf{Q}^{(75)}(L)$ ($\%\overline{L}$)	$\mathbf{Q}^{(90)}(L)$ ($\%\overline{L}$)	$\mathbf{Q}^{(95)}(L)$ ($\%\overline{L}$)
1	1	3.51	102.37	104.52	105.85
2	1	4.53	103.05	105.81	107.35
3	1	4.70	103.18	106.03	107.78
4	1	4.77	103.05	106.26	108.25
5	2	4.87	103.28	106.36	108.19
6	2	5.08	103.41	106.74	108.51
7	1	5.19	103.40	106.71	108.67
8	1	6.10	103.95	107.80	110.22
9	2	6.22	104.11	107.94	110.63
10	1	6.50	104.28	108.36	110.92
11	1	6.82	104.18	109.08	112.18
12	2	6.97	104.67	109.18	111.89
13	2	7.35	104.90	109.74	112.48
14	2	7.59	105.09	109.86	112.78
15	2	7.64	105.25	109.82	112.55
16	2	7.66	105.10	109.79	112.73
17	2	7.75	104.95	110.24	113.46
18	3	8.04	105.28	110.33	113.48
19	2	8.85	105.62	111.73	115.67
20	3	9.34	106.12	112.22	115.91
21	2	9.42	105.88	112.45	116.74
22	2	9.56	106.12	112.30	116.40
23	2	9.87	106.60	112.86	116.36
24	2	10.03	106.50	113.25	117.03
25	3	10.14	106.26	113.39	117.76
26	3	10.71	106.88	114.20	118.61
27	3	10.89	106.96	113.99	118.42
28	3	11.69	107.24	115.31	120.78
29	3	11.77	107.50	114.88	119.37
30	3	12.07	107.74	115.62	120.34
31	3	12.36	107.41	115.54	120.99
32	3	12.94	108.67	117.03	122.57
33	4	14.34	109.17	118.48	124.53
34	1	14.37	108.75	119.47	126.44
35	2	14.98	109.06	118.02	124.40
36	3	15.34	109.88	119.83	125.68
37	3	15.90	110.25	120.85	127.75
38	4	19.25	112.10	125.44	134.30
39	4	20.18	111.91	126.71	136.26
40	3	24.17	114.97	132.41	143.52
average		9.99	106.38	113.01	117.19

Table 4.1: ODP model – Summary statistics of the predictive distribution of L

CodCv	Class	$\mathbf{Cv}(L)$ (%)	$\mathbf{Q}^{(75)}(L)$ ($\%\overline{L}$)	$\mathbf{Q}^{(90)}(L)$ ($\%\overline{L}$)	$\mathbf{Q}^{(95)}(L)$ ($\%\overline{L}$)
1	1	2.59	101.73	103.34	104.32
2	1	4.00	102.65	105.17	106.71
3	1	4.26	102.82	105.51	107.16
4	1	3.68	102.45	104.76	106.17
5	2	3.40	102.26	104.39	105.68
6	2	3.46	102.30	104.47	105.79
7	1	3.99	102.65	105.17	106.70
8	1	4.70	103.11	106.09	107.92
9	2	4.96	103.27	106.42	108.36
10	1	4.74	103.13	106.15	107.99
11	1	4.36	102.88	105.64	107.32
12	2	5.63	103.71	107.31	109.53
13	2	5.69	103.74	107.39	109.63
14	2	4.81	103.17	106.23	108.10
15	2	5.79	103.81	107.52	109.81
16	2	6.02	103.95	107.82	110.20
17	2	8.50	105.51	111.08	114.56
18	3	5.37	103.54	106.97	109.08
19	2	5.58	103.67	107.24	109.44
20	3	5.55	103.65	107.20	109.39
21	2	5.86	103.85	107.61	109.92
22	2	6.20	104.07	108.05	110.51
23	2	7.27	104.74	109.46	112.39
24	2	8.21	105.33	110.71	114.05
25	3	6.39	104.19	108.30	110.84
26	3	9.55	106.15	112.48	116.44
27	3	9.19	105.93	112.00	115.79
28	3	9.64	106.21	112.60	116.60
29	3	6.67	104.37	108.68	111.34
30	3	9.83	106.32	112.84	116.94
31	3	7.47	104.87	109.73	112.74
32	3	7.62	104.96	109.93	113.01
33	4	7.74	105.03	110.08	113.21
34	1	8.47	105.49	111.05	114.52
35	2	13.97	108.77	118.35	124.48
36	3	16.73	110.33	122.03	129.63
37	3	19.94	112.04	126.31	135.71
38	4	12.36	107.84	116.21	121.52
39	4	9.96	106.41	113.02	117.18
40	3	10.66	106.83	113.94	118.43
average		7.27	104.69	109.48	112.48

Table 4.2: DFCL model – Summary statistics of the predictive distribution of L (lognormal assumption)

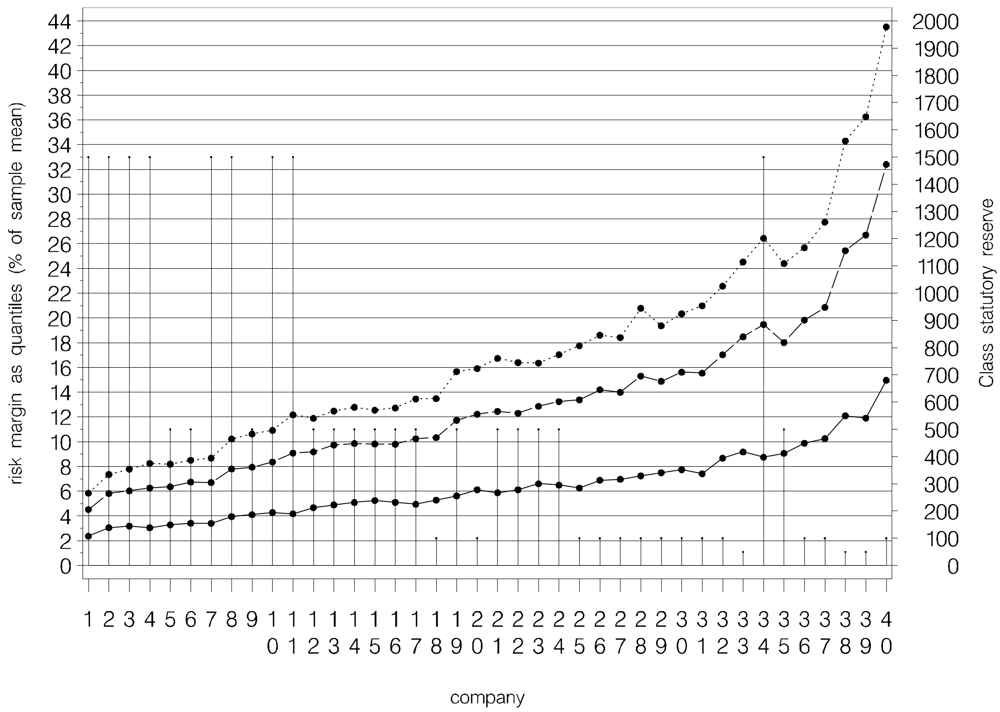

Figure 4.1: Risk margins from percentiles under the ODP model

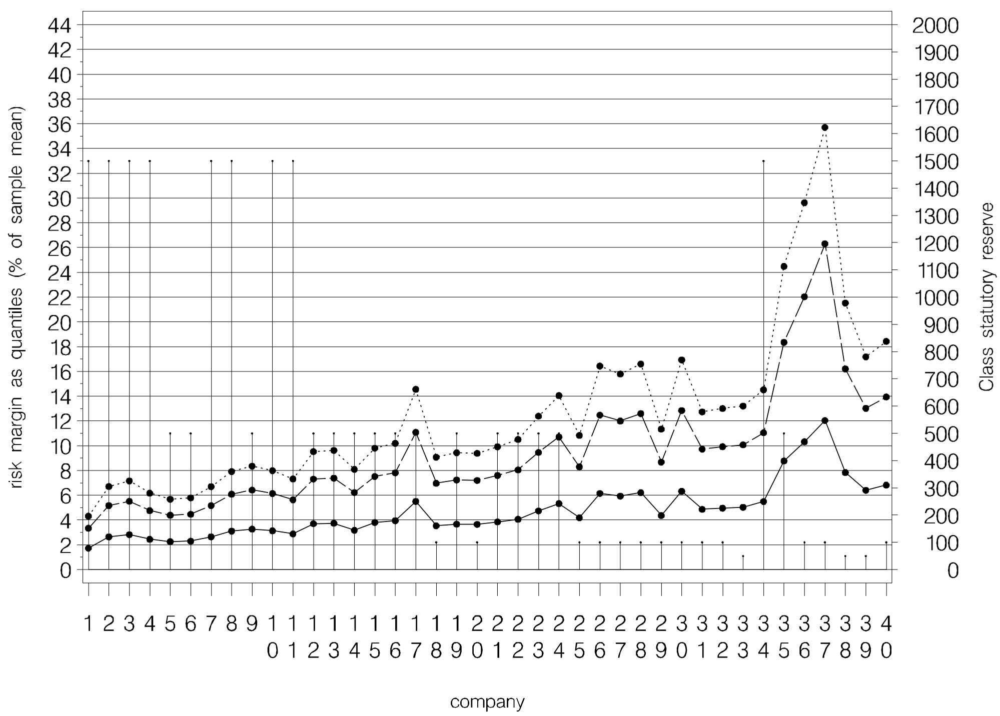

Figure 4.2: Risk margins from percentiles under the DFCL model (lognormal assumption)

4.1.3 OLL variability under the two models

As concerning the OLL variability a comparison between the two models is provided in table 4.3 where we reported the relative differences (in percent) between the values of $\mathbf{Cv}$, defined as:

$$\Delta\mathbf{Cv} := \frac{\mathbf{Cv}_{\mathrm{DFCL}}(L) - \mathbf{Cv}_{\mathrm{ODP}}(L)}{\mathbf{Cv}_{\mathrm{ODP}}(L)} .$$

The values of $\boldsymbol{\Delta}\mathbf{Cv}$ are negative for nearly all of the companies in the sample, indicating that the ODP model is more conservative than the Mack's model in terms of variability. In the same table we also reported the relative differences (in percent) between quantiles:

$$\Delta\mathbf{Q}^{(\alpha)} := \frac{\mathbf{Q}^{(\alpha)}_{\mathrm{DFCL}}(L) - \mathbf{Q}^{(\alpha)}_{\mathrm{ODP}}(L)}{\mathbf{Q}^{(\alpha)}_{\mathrm{ODP}}(L)} ,$$

and the percentage differences between risk margins:

$$\Delta\mathrm{RM}^{(\alpha)} := \frac{\mathbf{Q}^{(\alpha)}_{\mathrm{DFCL}}(L) - \mathbf{Q}^{(\alpha)}_{\mathrm{ODP}}(L)}{\mathbf{Q}^{(\alpha)}_{\mathrm{ODP}}(L) - \overline{L}} ,$$

for $\alpha = 75,\ 90,\ 95\%$. Of course a comparison between quantiles is influenced by the additional lognormality assumption introduced in the Mack's model. However also for the coefficients of variation the differences are negative for almost all the companies.

In table 4.4 some elementary statistics on $\Delta\mathbf{Cv}$, $\Delta\mathbf{Q}^{(\alpha)}$ and $\Delta\mathrm{RM}^{(\alpha)}$ across the selected sample are reported.

CodCv	Class	**ΔCv** (%)	$\mathbf{\Delta Q}^{(75)}$ (%)	$\mathbf{\Delta Q}^{(90)}$ (%)	$\mathbf{\Delta Q}^{(95)}$ (%)	$\Delta\mathrm{RM}^{(75)}$ (%)	$\Delta\mathrm{RM}^{(90)}$ (%)	$\Delta\mathrm{RM}^{(95)}$ (%)
1	1	-26.22	-0.63	-1.13	-1.45	-27.17	-26.16	-26.17
2	1	-11.80	-0.39	-0.61	-0.60	-13.23	-11.06	-8.79
3	1	-9.29	-0.35	-0.48	-0.58	-11.26	-8.51	-8.02
4	1	-22.68	-0.59	-1.41	-1.92	-19.85	-23.94	-25.15
5	2	-30.24	-0.99	-1.85	-2.31	-31.24	-30.98	-30.57
6	2	-31.86	-1.08	-2.13	-2.50	-32.63	-33.69	-31.93
7	1	-23.06	-0.73	-1.44	-1.81	-22.13	-22.96	-22.72
8	1	-22.90	-0.81	-1.58	-2.09	-21.38	-21.88	-22.52
9	2	-20.33	-0.80	-1.41	-2.05	-20.37	-19.13	-21.37
10	1	-26.99	-1.10	-2.04	-2.64	-26.80	-26.46	-26.81
11	1	-36.10	-1.24	-3.16	-4.33	-31.01	-37.92	-39.87
12	2	-19.20	-0.93	-1.71	-2.11	-20.72	-20.34	-19.86
13	2	-22.59	-1.10	-2.14	-2.53	-23.61	-24.15	-22.84
14	2	-36.71	-1.82	-3.31	-4.16	-37.63	-36.84	-36.68
15	2	-24.21	-1.37	-2.09	-2.44	-27.47	-23.39	-21.85
16	2	-21.38	-1.09	-1.80	-2.24	-22.50	-20.20	-19.87
17	2	9.64	0.53	0.77	0.97	11.31	8.26	8.20
18	3	-33.20	-1.66	-3.05	-3.88	-33.00	-32.54	-32.68
19	2	-36.96	-1.84	-4.02	-5.39	-34.67	-38.29	-39.79
20	3	-40.52	-2.32	-4.47	-5.63	-40.30	-41.06	-40.99
21	2	-37.77	-1.91	-4.31	-5.84	-34.49	-38.92	-40.73
22	2	-35.16	-1.94	-3.78	-5.06	-33.56	-34.55	-35.91
23	2	-26.30	-1.75	-3.01	-3.41	-28.17	-26.43	-24.25
24	2	-18.14	-1.09	-2.24	-2.55	-17.95	-19.18	-17.50
25	3	-37.04	-1.96	-4.49	-5.87	-33.18	-37.99	-38.96
26	3	-10.83	-0.68	-1.51	-1.83	-10.61	-12.14	-11.66
27	3	-15.63	-0.96	-1.75	-2.22	-14.76	-14.24	-14.25
28	3	-17.53	-0.96	-2.35	-3.46	-14.24	-17.69	-20.11
29	3	-43.30	-2.91	-5.40	-6.73	-41.76	-41.70	-41.45
30	3	-18.61	-1.31	-2.40	-2.83	-18.28	-17.75	-16.74
31	3	-39.59	-2.37	-5.04	-6.81	-34.34	-37.43	-39.29
32	3	-41.08	-3.41	-6.07	-7.80	-42.75	-41.70	-42.36
33	4	-46.04	-3.79	-7.09	-9.09	-45.12	-45.46	-46.14
34	1	-41.04	-2.99	-7.05	-9.43	-37.21	-43.24	-45.09
35	2	-6.74	-0.26	0.29	0.07	-3.16	1.87	0.35
36	3	9.10	0.41	1.84	3.14	4.54	11.11	15.38
37	3	25.45	1.62	4.52	6.23	17.44	26.17	28.69
38	4	-35.79	-3.80	-7.36	-9.51	-35.21	-36.29	-37.24
39	4	-50.63	-4.92	-10.80	-14.00	-46.20	-51.23	-52.62
40	3	-55.91	-7.08	-13.95	-17.48	-54.40	-56.98	-57.66

Table 4.3: ODP vs DFCL model: percentage differences in variability measures (DFCL: lognormal assumption)

	minimum	maximum	average	std
$\Delta\mathbf{Cv}$	-55.91	25.45	-25.73	16.49
$\Delta\mathbf{Q}^{(75)}$	-7.68	1.07	-1.68	1.63
$\Delta\mathbf{Q}^{(90)}$	-14.43	3.99	-3.13	3.27
$\Delta\mathbf{Q}^{(95)}$	-17.91	5.73	-3.95	4.23
$\Delta\mathrm{RM}^{(75)}$	-59.41	12.31	-26.83	15.80
$\Delta\mathrm{RM}^{(90)}$	-59.29	23.26	-26.41	17.12
$\Delta\mathrm{RM}^{(95)}$	-59.38	26.50	-26.28	17.88

Table 4.4: ODP vs DFCL model: summary statistics of percentage differences in variability measure

4.1.4 Measures of reserve adequacy on undiscounted basis

Appropriate measures of reserve adequacy can be derived for each company in the sample by comparing the level R^* of the required reserve and the level R^s of the statutory reserve. Obviously the reserve adequacy depends on the particular definition of the required reserve R^*. Tables 4.5 and 4.6 report results on reserve adequacy assuming four different definitions of R^*: the sample mean $\overline{L}$ and the quantiles $\mathbf{Q}^{(75)}(L)$, $\mathbf{Q}^{(90)}(L)$ and $\mathbf{Q}^{(95)}(L)$ of the predictive distribution of the undiscounted OLL. The following relative adequacy measures are considered:

- the difference $\overline{\Delta}$ between the statutory reserve and the sample mean of the overall OLL, as a percent of the statutory reserve:

$$\overline{\Delta} := \frac{R^s - \overline{L}}{R^s}\,;$$

- the difference $\Delta^{(\alpha)}$ between the statutory reserve and the α-th quantile $\mathbf{Q}^{(\alpha)}(L)$ of the OLL distribution, as a percent of the statutory reserve, for $\alpha = 75, 90, 95\%$:

$$\Delta^{(\alpha)} := \frac{R^s - \mathbf{Q}^{(\alpha)}(L)}{R^s}\,.$$

The *critical probability*:

$$p^* := \mathbf{P}(L \le R^s)\,,$$

which can be considered as an alternative measure of reserve adequacy, is also reported. In these tables companies are ordered by decreasing values of $\overline{\Delta}$.

Elementary summary statistics for the two models are given in tables 4.7 and 4.8.

A graphical representation of data in table 4.5 and 4.6 is provided in figure 4.3 and 4.4, respectively. On the horizontal axis the code number CodCv of the companies, sorted by decreasing values of $\overline{\Delta}$, is reported. The values of $\overline{\Delta}$ are represented with dots joined by solid line, $\Delta^{(75)}$ with circles joined by dashed line, and $\Delta^{(90)}$ with diamonds joined by dotted line. The values of the three adequacy measures are reported on the left vertical axis. As usual, the conventional statutory reserves (1500, 500, 100 and 50) are also reported on the right vertical axis and represented by vertical bars.

Results on reserve adequacy expressed in terms of monetary amounts are reported in tables 4.9, 4.10, 4.11 and 4.12. The analysis is developed here at aggregate level, on the four

n	CodCv	Class	$\overline{\Delta}$ (%)	$\Delta^{(75)}$ (%)	$\Delta^{(90)}$ (%)	$\Delta^{(95)}$ (%)	p^*
1	30	3	52.18	48.48	44.71	42.46	1.00
2	8	1	38.31	35.88	33.50	32.01	1.00
3	34	1	25.38	18.85	10.85	5.65	0.98
4	36	3	24.26	16.78	9.24	4.82	0.98
5	10	1	23.11	19.82	16.69	14.72	1.00
6	11	1	20.06	16.72	12.80	10.32	1.00
7	22	2	17.84	12.81	7.73	4.37	0.98
8	20	3	17.76	12.73	7.71	4.68	0.99
9	24	2	16.47	11.05	5.41	2.25	0.97
10	9	2	14.36	10.85	7.56	5.26	0.99
11	31	3	13.48	7.07	0.04	-4.67	0.90
12	39	4	13.27	2.95	-9.89	-18.17	0.79
13	5	2	10.82	7.89	5.15	3.52	0.99
14	1	1	9.13	6.97	5.02	3.81	1.00
15	18	3	8.01	3.16	-1.49	-4.39	0.86
16	29	3	6.29	-0.74	-7.66	-11.86	0.72
17	27	3	3.66	-3.04	-9.82	-14.08	0.67
18	40	3	1.38	-13.38	-30.58	-41.54	0.54
19	2	1	0.50	-2.54	-5.29	-6.82	0.53
20	15	2	-0.03	-5.28	-9.85	-12.58	0.48
21	13	2	-0.27	-5.19	-10.04	-12.79	0.52
22	6	2	-0.77	-4.21	-7.56	-9.34	0.45
23	19	2	-4.81	-10.70	-17.11	-21.23	0.30
24	4	1	-5.44	-8.66	-12.04	-14.14	0.14
25	35	2	-6.74	-16.41	-25.97	-32.79	0.31
26	14	2	-6.93	-12.37	-17.47	-20.60	0.21
27	16	2	-7.56	-13.04	-18.09	-21.25	0.19
28	33	4	-10.01	-20.10	-30.34	-36.99	0.27
29	3	1	-10.94	-14.47	-17.63	-19.58	0.02
30	32	3	-18.63	-28.92	-38.83	-45.41	0.10
31	12	2	-18.85	-24.40	-29.76	-32.98	0.01
32	7	1	-19.20	-23.26	-27.20	-29.54	0.00
33	21	2	-20.45	-27.53	-35.45	-40.61	0.02
34	37	3	-21.20	-33.63	-46.48	-54.83	0.12
35	28	3	-22.56	-31.43	-41.31	-48.02	0.04
36	38	4	-23.65	-38.61	-55.10	-66.06	0.16
37	23	2	-28.35	-36.83	-44.87	-49.35	0.01
38	25	3	-36.93	-45.50	-55.25	-61.24	0.00
39	17	2	-43.09	-50.17	-57.74	-62.35	0.00
40	26	3	-63.16	-74.40	-86.33	-93.52	0.00

Table 4.5: Reserve adequacy measures under the ODP model

n	CodCv	Class	$\overline{\Delta}$ (%)	$\Delta^{(75)}$ (%)	$\Delta^{(90)}$ (%)	$\Delta^{(95)}$ (%)	p^*
1	30	3	52.18	49.16	46.04	44.08	1.00
2	8	1	38.31	36.40	34.56	33.43	1.00
3	34	1	25.38	21.28	17.14	14.55	1.00
4	36	3	24.26	16.44	7.58	1.83	0.96
5	10	1	23.11	20.70	18.39	16.97	1.00
6	11	1	20.06	17.75	15.55	14.20	1.00
7	22	2	17.84	14.50	11.22	9.20	1.00
8	20	3	17.76	14.76	11.84	10.04	1.00
9	24	2	16.47	12.02	7.53	4.73	0.99
10	9	2	14.36	11.56	8.86	7.21	1.00
11	31	3	13.48	9.27	5.07	2.46	0.98
12	39	4	13.27	7.72	1.98	-1.62	0.93
13	5	2	10.82	8.81	6.91	5.75	1.00
14	1	1	9.13	7.56	6.09	5.20	1.00
15	18	3	8.01	4.76	1.60	-0.34	0.94
16	29	3	6.29	2.20	-1.84	-4.34	0.84
17	27	3	3.66	-2.05	-7.90	-11.55	0.67
18	40	3	1.38	-5.35	-12.37	-16.79	0.57
19	2	1	0.50	-2.14	-4.65	-6.18	0.56
20	15	2	-0.03	-3.84	-7.55	-9.84	0.51
21	13	2	-0.27	-4.03	-7.68	-9.93	0.49
22	6	2	-0.77	-3.08	-5.27	-6.60	0.42
23	19	2	-4.81	-8.66	-12.40	-14.70	0.21
24	4	1	-5.44	-8.02	-10.46	-11.95	0.08
25	35	2	-6.74	-16.11	-26.33	-32.88	0.34
26	14	2	-6.93	-10.32	-13.59	-15.58	0.09
27	16	2	-7.56	-11.81	-15.97	-18.53	0.12
28	33	4	-10.01	-15.55	-21.10	-24.54	0.12
29	3	1	-10.94	-14.07	-17.06	-18.88	0.01
30	32	3	-18.63	-24.52	-30.41	-34.07	0.01
31	12	2	-18.85	-23.25	-27.54	-30.17	0.00
32	7	1	-19.20	-22.36	-25.36	-27.19	0.00
33	21	2	-20.45	-25.09	-29.61	-32.40	0.00
34	37	3	-21.20	-35.80	-53.09	-64.48	0.19
35	28	3	-22.56	-30.17	-38.00	-42.90	0.02
36	38	4	-23.65	-33.34	-43.69	-50.26	0.05
37	23	2	-28.35	-34.44	-40.50	-44.26	0.00
38	25	3	-36.93	-42.66	-48.29	-51.77	0.00
39	17	2	-43.09	-50.97	-58.95	-63.93	0.00
40	26	3	-63.16	-73.20	-83.52	-89.98	0.00

Table 4.6: Reserve adequacy measures under the DFCL model

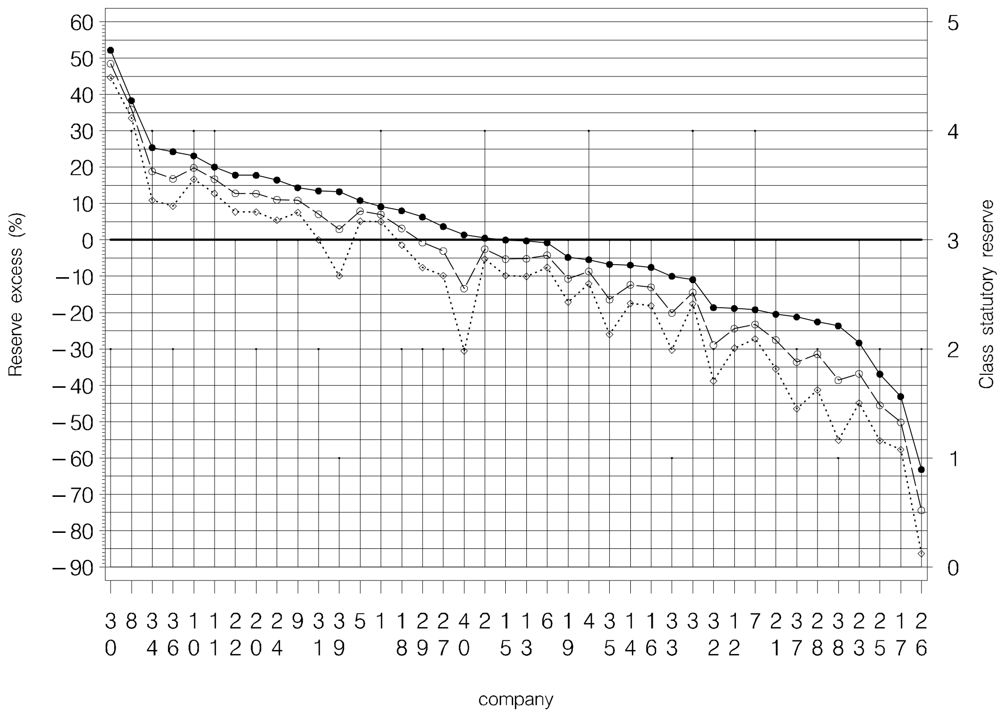

Figure 4.3: Reserve adequacy under the ODP model

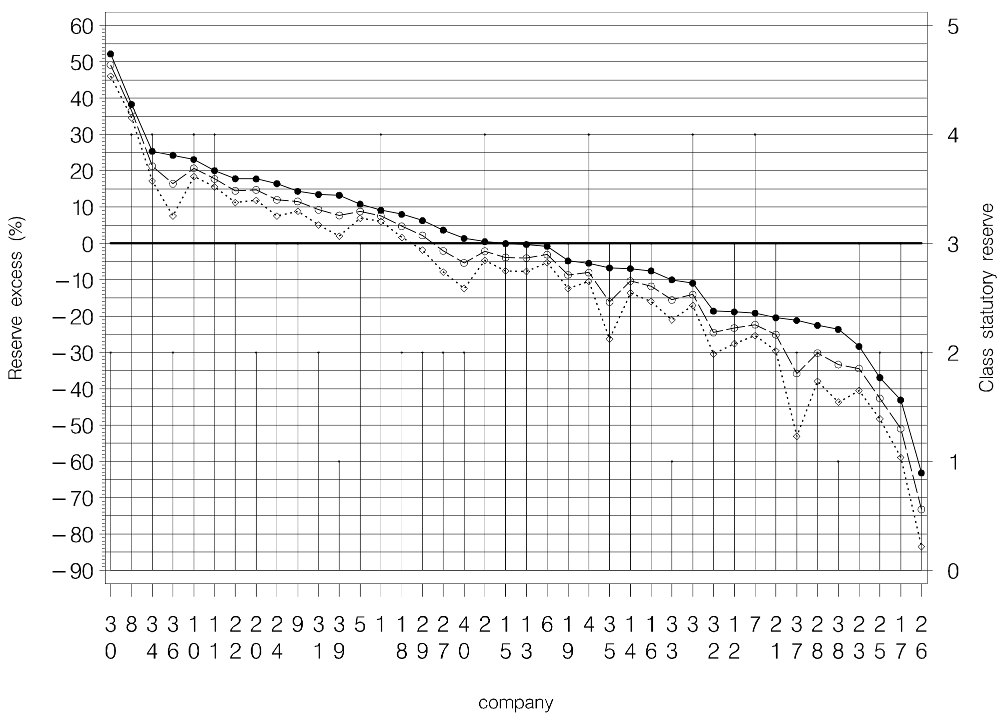

Figure 4.4: Reserve adequacy under the DFCL model

	minimum	maximum	average	std
$\overline{\Delta}$	-63.16	52.18	-1.33	22.37
$\Delta^{(75)}$	-74.39	48.48	-7.82	24.14
$\Delta^{(90)}$	-86.33	44.71	-14.57	26.37
$\Delta^{(95)}$	-94.52	42.46	-18.82	27.99
$\mathbf{Cv}(L)$	3.51	24.17	9.99	4.64

Table 4.7: ODP model – Summary statistics on reserve adequacy measures

	minimum	maximum	average	std
$\overline{\Delta}$	-63.16	52.18	-1.33	22.37
$\Delta^{(75)}$	-75.20	49.16	-6.15	23.97
$\Delta^{(90)}$	-83.52	46.04	-11.07	25.91
$\Delta^{(95)}$	-89.98	44.08	-14.15	27.26
$\mathbf{Cv}(L)$	2.59	19.94	7.27	3.63

Table 4.8: DFCL model – Summary statistics on reserve adequacy measures

dimensional classes and on the overall sample. Table 4.9 illustrates the required reserve levels (in million Euros) defined by posing $R^* = \overline{L}$ and $R^* = \mathbf{Q}^{(\alpha)}(L)$, with $\alpha = 75, 90, 95\%$. The number $N_{\overline{L}}$ and N_α of companies having a corresponding non-negative excess reserve, that is such that $R^s - R^* \geq 0$ (for $R^* = \overline{L}$ and $R^* = \mathbf{Q}^{(\alpha)}$), is also indicated. Table 4.10 contains the analogous results referred to the Mack's model.

The same informations, expressed in terms of excess of reserve $R^s - R^*$ (in million Euros) are reported in tables 4.11 and 4.12. The values of $\overline{\Delta} := R^s - \overline{L}$ and $\Delta^{(\alpha)} := R^s - \mathbf{Q}^{(\alpha)}(L)$, for $\alpha = 75, 90, 95\%$, are given for the two models.

4.1.5 Risk margins as the cost of capital

Alternative measures of reserve adequacy can be introduced using risk margins defined as the cost of the reserve risk capital. This point of view will be assumed in this paragraph, only considering for the moment, the undiscounted case under the ODP model.

Reserve risk capital and the cost of risk capital

In our application K_0 denotes the reserve risk capital for the year 2005, that is the solvency capital required at the end of 2004 and to be maintained until the end of 2005. In section 1.2.6 several alternative approaches to risk capital computation have been considered. In section 1.3 these different methods have been adapted to obtain consistency if the corresponding cost of capital κ_0 is used for determining the reserve risk margin λ_0. At this stage we use the YEE undiscounted method, providing the expression (1.71):

$$K_0 = \frac{\mathbf{W}_0(Z_1) - \overline{L}}{1+s},$$

Class	R^s	$\overline{L}$	$N_{\overline{L}}$	$\mathbf{Q}^{(75)}$	N_{75}	$\mathbf{Q}^{(90)}$	N_{90}	$\mathbf{Q}^{(95)}$	N_{95}
1	15,114.43	14,116.69	6	14,662.96	5	15,232.53	5	15,586.90	5
2	7,257.39	7,604.49	4	8,002.60	4	8,397.04	4	8,641.54	4
3	1,541.70	1,556.26	8	1,679.02	5	1,807.45	4	1,888.57	3
4	128.45	135.65	1	150.46	1	167.04	0	177.87	0
total	24,041.97	23,413.09	19	24,495.04	15	25,604.06	13	26,294.87	12

Table 4.9: ODP model – Required reserves by classes

Class	R^s	$\overline{L}$	$N_{\overline{L}}$	$\mathbf{Q}^{(75)}$	N_{75}	$\mathbf{Q}^{(90)}$	N_{90}	$\mathbf{Q}^{(95)}$	N_{95}
1	15,114.43	14,116.69	6	14,531.09	5	14,930.39	5	15,175.18	5
2	7,257.39	7,604.49	4	7,910.61	4	8,214.97	4	8,403.65	4
3	1,541.70	1,556.26	8	1,641.22	6	1,729.03	5	1,784.24	4
4	128.45	135.65	1	144.26	1	153.20	1	158.82	0
total	24,041.97	23,413.09	19	24,227.17	16	25,027.59	15	25,521.89	13

Table 4.10: DFCL model – Required reserves by classes

Class	$\overline{\Delta}$	$(\%R^s)$	$\Delta^{(75)}$	$(\%R^s)$	$\Delta^{(90)}$	$(\%R^s)$	$\Delta^{(95)}$	$(\%R^s)$
1	997.74	6.60	451.47	2.99	-118.10	-0.78	-472.47	-3.13
2	-347.10	-4.78	-745.21	-10.27	-1,139.65	-15.70	-1,384.15	-19.07
3	-14.56	-0.94	-137.32	-8.91	-265.75	-17.24	-346.87	-22.50
4	-7.20	-5.60	-22.01	-17.14	-38.59	-30.04	-49.42	-38.47
total	628.88	2.62	-453.07	-1.88	-1,562.09	-6.50	-2,252.90	-9.37

Table 4.11: ODP model – Excesses of reserve by classes

Class	$\overline{\Delta}$	$(\%R^s)$	$\Delta^{(75)}$	$(\%R^s)$	$\Delta^{(790)}$	$(\%R^s)$	$\Delta^{(95)}$	$(\%R^s)$
1	997.74	6.60	583.34	3.86	184.035	1.22	-60.75	-0.40
2	-347.10	-4.78	-653.22	-9.00	-957.578	-13.19	-1,146.26	-15.79
3	-14.56	-0.94	-99.52	-6.46	-187.332	-12.15	-242.53	-15.73
4	-7.20	-5.60	-15.81	-12.31	-24.746	-19.27	-30.37	-23.64
total	628.88	2.62	-185.20	-0.77	-985.621	-4.10	-1,479.92	-6.16

Table 4.12: DFCL model – Excesses of reserve by classes

where $\mathbf{W}_0(Z_1)$ is the RAV of the year-end obligations:

$$Z_1 = Y_1 + \sum_{\tau=2}^{n-1} \mathbf{E}_1(Y_\tau)\,,$$

and s is the positive spread expressing the excess return required by the shareholders over the risk-free rate for investing in the insurance business (see also table 1.3). The corresponding risk margin provided by expression (1.72) is:

$$\lambda_0 = \frac{s}{1+s}\,\frac{\mathbf{W}_0(Z_1) - \overline{L}}{\overline{L}} \sum_{\tau=1}^{T}\sum_{\theta=\tau}^{T} \overline{Y}_\theta\,,$$

which is equal to the cost of capital κ_0 by construction.

We derived the predictive distribution of Z_1 for each company in the selected sample under the ODP model, using a bootstrap procedure analogous to that introduced in [13]. The RAV $\mathbf{W}_0(Z_1)$ has been specified as the quantile at the confidence level $\alpha = 99.5\%$ and has been computed on this distribution. The spread has been posed at the level $s = 6\%$ (as in the Swiss Solvency Test) and the corresponding value of K_0 has been calculated.

We also computed the risk capital K_0' specifying the worst case $\mathbf{W}_0(Z_1)$ as the expected shortfall at the 99% confidence level, with the same value of s.

The values of K_0 expressed as a percent of the sample mean $\overline{L}$ are illustrated in figure 4.5 using dots joined by solid line. The corresponding values of K_0' are represented by circles joined by dashed line. The companies, reported on the horizontal axis, are sorted by increasing value of CodCv; the usual representation of the conventional statutory reserve is also provided. The numerical values of K_0 and K_0' (as a percent of $\overline{L}$) are reported in table 4.13. In the same table we also provide the corresponding values κ_0 and κ_0' of the cost of capital, expressed as a percent of $\overline{L}$; κ_0 is expressed also as a percent of the K_0 reserve risk capital (the ratio between κ_0' and K_0' has the same value). These costs are graphically illustrated in figure 4.6, with the same symbols of figure 4.5.

Comparison with the percentile approach

The comparison between risk margins derived as quantiles and risk margins computed as costs of capital is provided in figure 4.7. As in figure 4.1 the risk margins are expressed as a percentage of the sample mean $\overline{L}$ and the companies are sorted by increasing **Cv**. Risk margins as the 75-th and 90-th quantile of L are reported, represented by diamonds joined by solid line and by dashed line, respectively. Dots joined by solid line and by dashed line represent risk margins determined by the cost of capital κ_0 and κ_0', respectively.

In table 4.14 the comparison is provided between risk margins by 75% quantile and risk margins as cost of capital (as 99.5% quantile). As in tables 4.9 and 4.11 the analysis is made in terms of reserve adequacy for the four dimensional classes.

CodCv	Class	K_0 $(\%\overline{L})$	K_0' $(\%\overline{L})$	κ_0 $(\%K_0)$	κ_0 $(\%\overline{L})$	κ_0' $(\%\overline{L})$
1	1	9.69	10.08	17.22	1.67	1.74
2	1	12.69	13.25	15.95	2.02	2.11
3	1	12.66	13.14	17.20	2.18	2.26
4	1	13.52	14.04	17.77	2.40	2.50
5	2	14.16	14.35	14.43	2.04	2.07
6	2	14.94	15.62	18.64	2.79	2.91
7	1	15.34	16.19	17.77	2.73	2.88
8	1	18.28	18.86	14.61	2.67	2.76
9	2	17.62	18.37	17.71	3.12	3.25
10	1	17.89	18.80	17.45	3.12	3.28
11	1	21.85	22.64	17.53	3.83	3.97
12	2	18.89	20.03	15.03	2.84	3.01
13	2	21.91	22.64	17.85	3.91	4.04
14	2	22.02	23.05	18.30	4.03	4.22
15	2	21.04	22.17	15.25	3.21	3.38
16	2	20.70	21.84	17.21	3.56	3.76
17	2	23.62	24.92	19.76	4.67	4.92
18	3	24.13	25.41	17.12	4.13	4.35
19	2	29.98	31.62	19.31	5.79	6.11
20	3	28.68	29.21	18.39	5.28	5.37
21	2	30.09	31.60	19.07	5.74	6.03
22	2	28.09	28.94	16.87	4.74	4.88
23	2	28.86	29.21	15.49	4.47	4.52
24	2	29.99	31.12	16.36	4.91	5.09
25	3	35.32	36.69	18.97	6.70	6.96
26	3	32.22	34.61	15.86	5.11	5.49
27	3	33.99	35.60	16.03	5.45	5.71
28	3	36.27	39.28	20.79	7.54	8.17
29	3	38.93	40.55	16.76	6.52	6.80
30	3	36.35	38.76	15.70	5.71	6.09
31	3	44.17	46.71	18.31	8.09	8.55
32	3	43.80	46.43	17.65	7.73	8.19
33	4	48.75	50.59	18.68	9.11	9.45
34	1	52.69	54.93	19.21	10.12	10.55
35	2	47.41	50.69	16.72	7.93	8.48
36	3	51.61	54.60	16.63	8.58	9.08
37	3	50.31	53.54	17.30	8.70	9.26
38	4	62.95	66.28	16.71	10.52	11.08
39	4	79.62	85.10	17.59	14.01	14.97
40	3	87.62	91.38	17.90	15.68	16.35

Table 4.13: Risk capitals and costs of capital under the ODP model

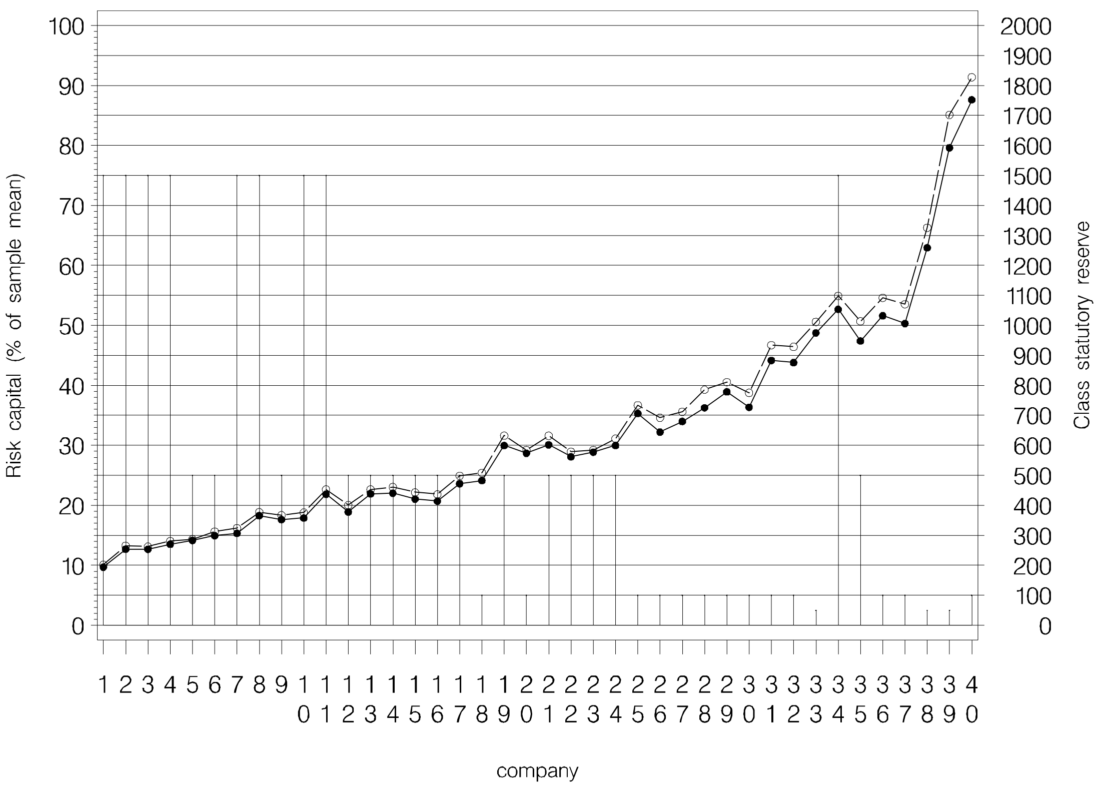

Figure 4.5: ODP model – Risk capitals as 99.5% quantiles and as 99% expected shortfall

Class	R^s	$\overline{L}$	$\mathbf{Q}^{(75)}$	$\Delta^{(75)}$	$(\%R^s)$	$\overline{L}+\kappa_0$	$\Delta^{(\mathrm{CoC})}$	$(\%R^s)$
1	15,114.43	14,116.69	14,662.96	451.47	2.99	14,579.29	535.14	3.54
2	7,257.39	7,604.49	8,002.60	-745.21	-10.27	7,920.16	-662.77	-9.13
3	1,541.70	1,556.26	1,679.02	-137.32	-8.91	1,668.34	-126.64	-8.21
4	128.45	135.65	150.46	-22.01	-17.14	150.56	-22.11	-17.21
total	24,041.97	23,413.09	24,495.04	-453.07	-1.88	24,318.35	-276.38	-1.15

Table 4.14: ODP model – Required reserves and excesses of reserve by classes

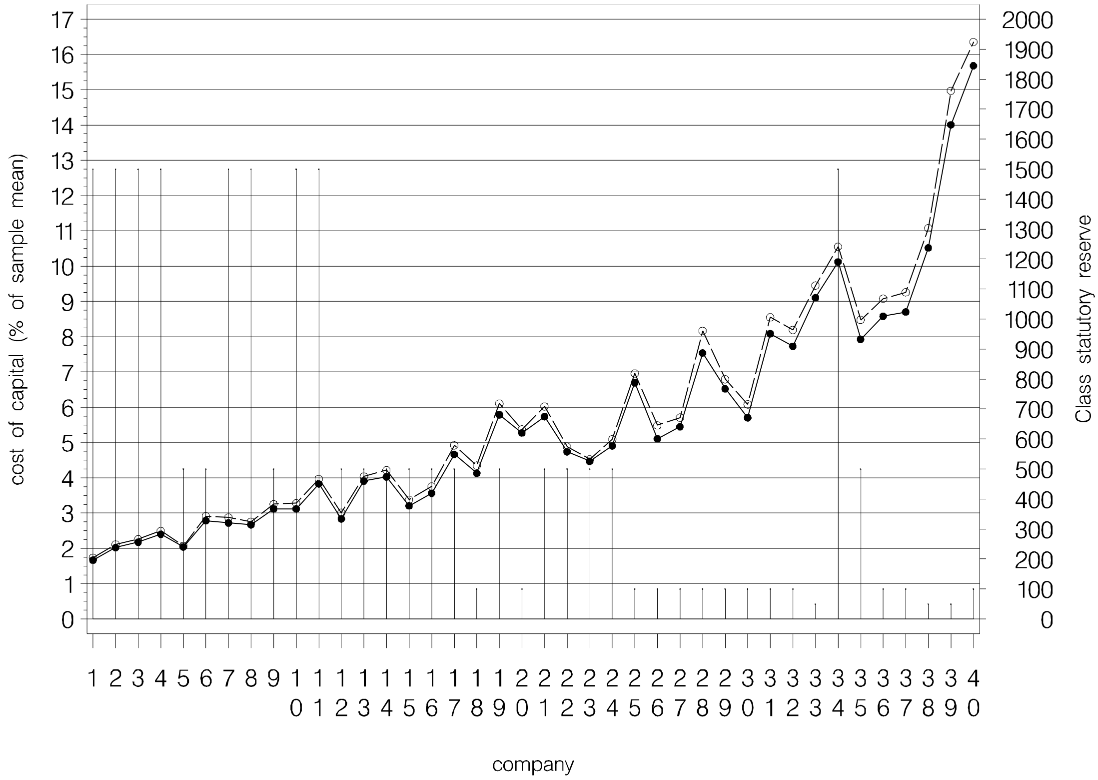

Figure 4.6: Risk margins as costs of capital - Spread to shareholders: 6%

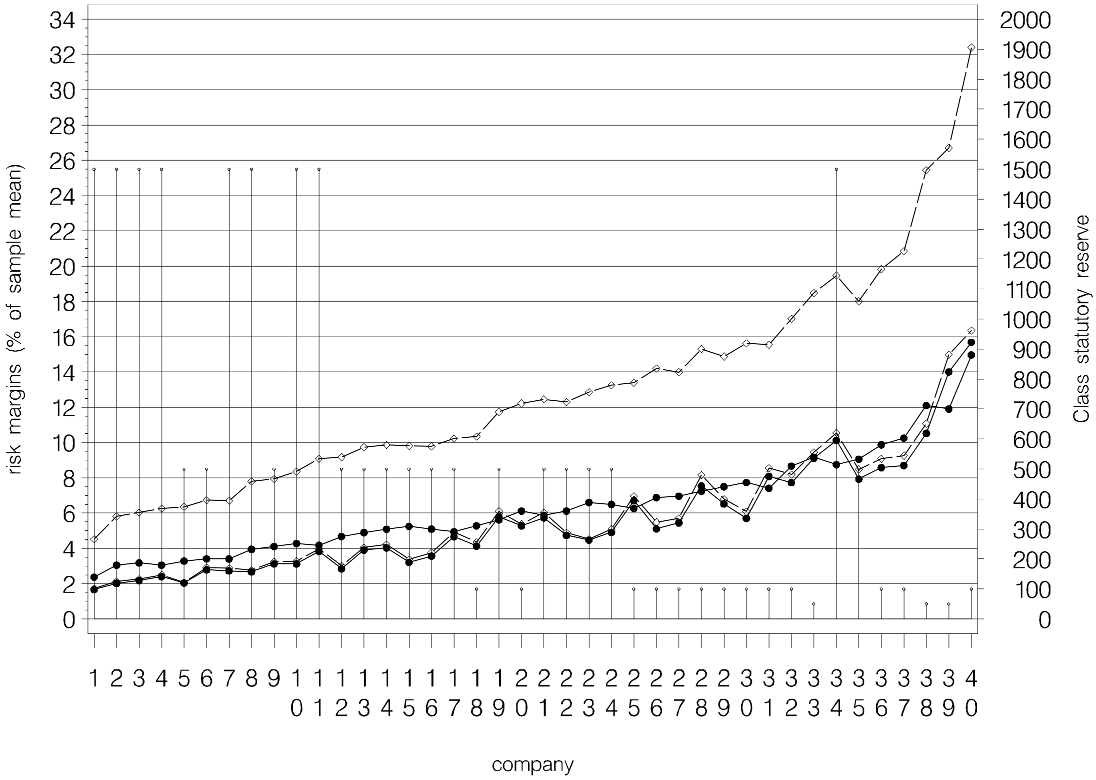

Figure 4.7: Risk margins as quantiles and as costs of capital

4.2 Discounted values

In order to provide a conservative assessment of the required reserve, the overall OLL are usually derived as the sum of the undiscounted future paid losses. However if an appropriate risk margin is used discounting could be viable, since the risk-loaded expected payoffs could be considered as the certainty equivalent of random liabilities.

To gain some insight into this issue we considered the predictive distribution of the r.v.:

$$D := \sum_{i=1}^{n} \sum_{\tau=1}^{n-1} v_\tau \, C_{i,d_i+\tau} ,$$

obtained by aggregating the discounted paid losses of the future triangle produced by the simulated ODP model. The term structure of risk-free interest rates prevailing on the market at the valuation date on December 31, 2004 was used for discounting. The discount factors v_τ and the corresponding annual interest rates i_τ for the relevant maturities $\tau = 1, 2, \ldots, 9$ are illustrated in table 4.15.

τ	v_τ	i_τ
1	0.9777	2.28
2	0.9507	2.56
3	0.9204	2.80
4	0.8879	3.02
5	0.8542	3.20
6	0.8200	3.36
7	0.7857	3.50
8	0.7519	3.63
9	0.7187	3.74

Table 4.15: The term structure of risk-free interest rates on December 31, 2004

Discounted mean and quantiles

On the empirical distribution of the discounted paid losses the α-quantiles $\mathbf{Q}^{(\alpha)}(D)$ have been computed for $\alpha = 75, 90, 95\%$. These quantiles are reported in table 4.16, expressed as a percent of $\overline{L}$. The corresponding value of the discounted sample mean:

$$M_0 := \sum_{i=1}^{n} \sum_{\tau=1}^{n-1} v_\tau \, \mathbf{E}_0 \left(C_{i,d_i+\tau} \right) = \sum_{\tau=1}^{n-1} v_\tau \, \overline{Y}_\tau ,$$

is also given in the table. Also in this case companies are sorted by CodCv. In table 4.17 we also provided the percentage differences between discounted and undiscounted values:

$$\Delta \overline{L}_v := \frac{M_0 - \overline{L}}{\overline{L}} .$$

and:

$$\Delta \mathbf{Q}_v^{(\alpha)} := \frac{\mathbf{Q}^{(\alpha)}(D) - \mathbf{Q}^{(\alpha)} L)}{\mathbf{Q}^{(\alpha)}(L)} .$$

CodCv	Class	M_0 ($\%\overline{L}$)	$\mathbf{Q}^{(75)}(D)$ ($\%\overline{L}$)	$\mathbf{Q}^{(90)}(D)$ ($\%\overline{L}$)	$\mathbf{Q}^{(95)}(D)$ ($\%\overline{L}$)
1	1	92.07	94.18	96.09	97.25
2	1	92.71	95.44	97.86	99.30
3	1	92.06	94.88	97.38	99.00
4	1	91.77	94.48	97.27	99.02
5	2	93.52	96.47	99.24	100.92
6	2	91.27	94.26	97.10	98.66
7	1	91.79	94.83	97.76	99.50
8	1	93.42	97.02	100.47	102.65
9	2	91.79	95.39	98.79	101.14
10	1	91.93	95.78	99.33	101.61
11	1	91.87	95.59	99.85	102.53
12	2	93.22	97.44	101.52	103.94
13	2	91.72	95.99	100.14	102.52
14	2	91.49	95.91	100.19	102.68
15	2	93.10	97.78	101.97	104.40
16	2	92.08	96.59	100.78	103.38
17	2	90.69	94.96	99.54	102.26
18	3	92.11	96.76	101.27	104.04
19	2	90.94	95.79	101.08	104.44
20	3	91.44	96.79	102.09	105.25
21	2	91.09	96.29	101.94	105.62
22	2	92.23	97.78	103.19	106.76
23	2	93.01	98.92	104.58	107.59
24	2	92.49	98.23	104.26	107.51
25	3	91.11	96.60	102.62	106.33
26	3	92.77	98.99	105.48	109.20
27	3	92.69	98.90	105.13	109.09
28	3	90.15	96.45	103.39	108.10
29	3	92.31	99.06	105.64	109.64
30	3	92.87	99.92	107.05	111.33
31	3	91.47	98.04	105.14	110.05
32	3	91.82	99.27	106.37	111.10
33	4	91.27	99.48	107.62	113.10
34	1	90.97	98.63	107.86	114.03
35	2	92.29	100.46	108.38	114.02
36	3	92.35	101.28	110.05	115.34
37	3	91.99	101.03	110.47	116.54
38	4	92.36	103.15	114.98	122.67
39	4	91.77	102.10	114.56	122.47
40	3	91.74	105.17	120.64	130.60

Table 4.16: Quantiles of discounted OLL in the ODP model

CodCv	Class	$\Delta\overline{L}_v$ ($\%\overline{L}$)	$\Delta\mathbf{Q}_v^{(75)}$ (%)	$\Delta\mathbf{Q}_v^{(90)}$ (%)	$\Delta\mathbf{Q}_v^{(95)}$ (%)
1	1	-7.93	-8.01	-8.07	-8.12
2	1	-7.29	-7.39	-7.52	-7.50
3	1	-7.94	-8.04	-8.16	-8.15
4	1	-8.23	-8.32	-8.47	-8.53
5	2	-6.48	-6.59	-6.69	-6.72
6	2	-8.73	-8.86	-9.03	-9.08
7	1	-8.21	-8.29	-8.38	-8.44
8	1	-6.58	-6.67	-6.80	-6.87
9	2	-8.21	-8.38	-8.47	-8.58
10	1	-8.07	-8.16	-8.33	-8.39
11	1	-8.13	-8.24	-8.46	-8.60
12	2	-6.78	-6.91	-7.01	-7.11
13	2	-8.28	-8.49	-8.74	-8.85
14	2	-8.51	-8.73	-8.80	-8.96
15	2	-6.90	-7.10	-7.15	-7.24
16	2	-7.92	-8.09	-8.21	-8.29
17	2	-9.31	-9.51	-9.70	-9.86
18	3	-7.89	-8.09	-8.22	-8.32
19	2	-9.06	-9.30	-9.54	-9.71
20	3	-8.56	-8.80	-9.03	-9.19
21	2	-8.91	-9.06	-9.35	-9.52
22	2	-7.77	-7.86	-8.12	-8.28
23	2	-6.99	-7.21	-7.34	-7.54
24	2	-7.51	-7.76	-7.93	-8.14
25	3	-8.89	-9.10	-9.50	-9.71
26	3	-7.23	-7.38	-7.64	-7.93
27	3	-7.31	-7.53	-7.77	-7.88
28	3	-9.85	-10.1	-10.3	-10.5
29	3	-7.69	-7.85	-8.05	-8.15
30	3	-7.13	-7.26	-7.41	-7.49
31	3	-8.53	-8.72	-9.00	-9.04
32	3	-8.18	-8.65	-9.10	-9.35
33	4	-8.73	-8.87	-9.16	-9.18
34	1	-9.03	-9.30	-9.72	-9.81
35	2	-7.71	-7.88	-8.16	-8.34
36	3	-7.65	-7.82	-8.16	-8.23
37	3	-8.01	-8.37	-8.59	-8.77
38	4	-7.64	-7.98	-8.34	-8.65
39	4	-8.23	-8.76	-9.59	-10.1
40	3	-8.26	-8.52	-8.89	-9.01

Table 4.17: ODP model – Percentage differences between discounted and undiscounted values

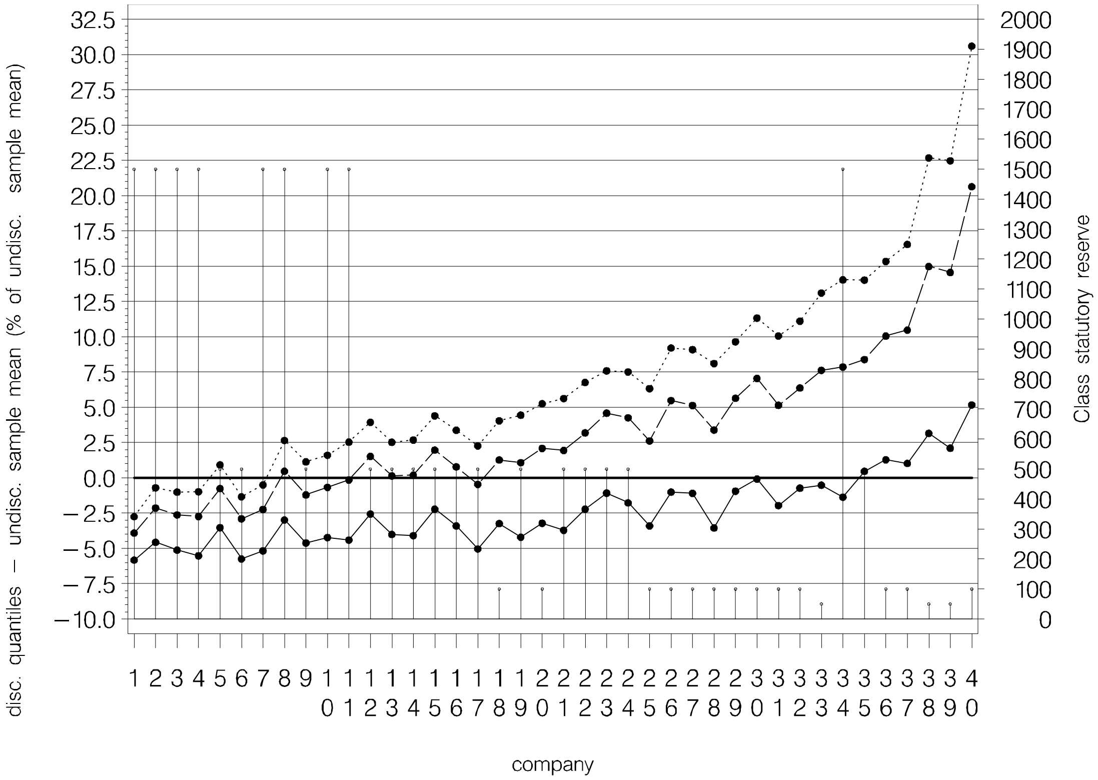

Figure 4.8: Differences between discounted quantiles and undiscounted means under the ODP model

The "discounted quantiles" $\mathbf{Q}^{(\alpha)}(D)$ can be considered as an alternative definition of the required reserve, and the difference with respect to the undiscounted sample mean $\overline{L}$ provides an indication of the corresponding reserve reduction. In figure 4.8 the percentage differences:

$$\frac{\mathbf{Q}^{(\alpha)}(D) - \overline{L}}{\overline{L}}$$

are illustrated for $\alpha = 75\%$ (solid line), $\alpha = 90\%$ (dashed line) and $\alpha = 95\%$ (dotted line).

Tables 4.18 and 4.19 illustrate results on reserve adequacy at aggregate level in the discounted case (for the ODP model). They are the analogous of tables 4.9 and 4.11, respectively, which were referred to the undiscounted case.

Approximated discounting

We also computed the approximated values $\mathbf{Q}^{(\alpha)}(D) \approx \varphi\, \mathbf{Q}^{(\alpha)}(L)$, obtained by rescaling the undiscounted sample statistics of the total OLL L by the correction factor:

$$\varphi := \frac{\sum_{\tau=1}^{n} v_\tau \overline{Y}_\tau}{\sum_{\tau=1}^{n} \overline{Y}_\tau} = \frac{M_0}{\overline{L}} . \tag{4.1}$$

As illustrated in section 1.2.6, φ is the weighted average of the discount factors v_τ weighted by the relative expected OLL. It captures the effect of the discounting on the sample means

$\overline{Y}_\tau$ and can be directly computed without an apposite run of the stochastic model. The percentage errors between approximated and non-approximated values are reported in table 4.20, together with the corresponding correction factors. The φ factor appears to provide slight overestimates of the correct values $\mathbf{Q}^{(\alpha)}(\widehat{D})$, with errors obviously increasing with α. However the approximation seems quite good, producing errors typically lower than 1% for the three values of α considered.

Risk margins as the costs of capital

Risk margins defined as the costs of capital have been computed also on discounted basis. Under the discounted YEE approach the risk capital is given by expression (1.63):

$$K_0 = \frac{v_1\,\mathbf{W}_0(M_1^-) - M_0}{1 + v_1\,s} = \frac{\mathbf{U}_0(v_1\,M_1^-)}{1 + v_1\,s}\,,$$

where the year-end obligations are:

$$M_1^- = Y_1 + \sum_{\tau=2}^{n-1} v_\tau\,\mathbf{E}_1(Y_\tau)\,.$$

The corresponding risk margin, which equals the cost of capital, is given by the expression (1.64):

$$\lambda_0 = \kappa_0 = s\,\frac{K_0}{M_0}\sum_{\tau=1}^{T} v_\tau\,\widehat{M}_{\tau-1} = \frac{s}{1+v_1\,s}\,\frac{v_1\,\mathbf{W}_0(M_1^-) - M_0}{M_0}\sum_{\tau=1}^{T}\frac{v_\tau}{v_{\tau-1}}\sum_{\theta=\tau}^{T} v_\theta\,\overline{Y}_\theta\,.$$

Also the predictive distribution of M_1^- has been derived under the ODP model applying the bootstrap procedure to each company in the selected sample. The RAV $\mathbf{W}_0(M_1^-)$ has been specified as the quantile at the security level $\alpha = 99.5\%$ and as the expected shortfall at the security level $\alpha = 99\%$; the corresponding risk capitals K_0 and K_0' and risk margins κ_0 and κ_0' have been derived assuming a spread at the level $s = 6\%$.

The numerical values of the risk capitals and of the risk margins are reported in table 4.21 expressed as a percent of the discounted sample mean M_0.

The comparison between risk margins derived as discounted quantiles and risk margins computed as costs of capital with the discounted YEE approach is provided in figure 4.9. To facilitate comparison with figure 4.7 all the risk margins have been expressed as a percentage of the undiscounted sample mean $\overline{L}$ and the same scale of figure 4.7 has been used. As usual the companies are sorted by increasing **Cv**, i.e. by the CodCv code. Risk margins as the 75-th and 90-th quantile of D are represented by diamonds joined by solid line and by dashed line, respectively. Dots joined by solid line and by dashed line represent risk margins determined by the cost of capital κ_0 and κ_0', respectively.

Class	R^s	M_0	N_{M_0}	$\mathbf{Q}^{(75)}(D)$	N_{75}	$\mathbf{Q}^{(90)}(D)$	N_{90}	$\mathbf{Q}^{(95)}(D)$	N_{95}
1	15,114.43	12,985.39	7	13,470.89	7	13,967.67	6	14,283.61	6
2	7,257.39	7,001.02	11	7,353.73	7	7,701.26	5	7,914.76	5
3	1,541.70	1,429.11	8	1,537.57	7	1,649.87	6	1,720.88	5
4	128.45	124.49	1	137.61	1	151.98	1	161.35	0
total	24,041.97	21,540.01	27	22,499.79	22	23,470.78	18	24,080.60	16

Table 4.18: ODP model– Required reserve by classes on discounted basis

Class	$\overline{\Delta}(D)$	$(\%R^s)$	$\Delta^{(75)}(D)$	$(\%R^s)$	$\Delta^{(90)}(D)$	$(\%R^s)$	$\Delta^{(95)}(D)$	$(\%R^s)$
1	2,129.04	14.09	1,643.5	10.87	1,146.76	7.59	830.82	5.50
2	256.37	3.53	-96.34	-1.33	-443.87	-6.12	-657.37	-9.06
3	112.60	7.30	4.13	0.27	-108.17	-7.02	-179.18	-11.62
4	3.96	3.08	-9.16	-7.13	-23.53	-18.32	-32.90	-25.61
total	2,501.96	10.41	1,542.18	6.41	571.19	2.38	-38.63	-0.16

Table 4.19: ODP model – Excesses of reserve by classes on discounted basis

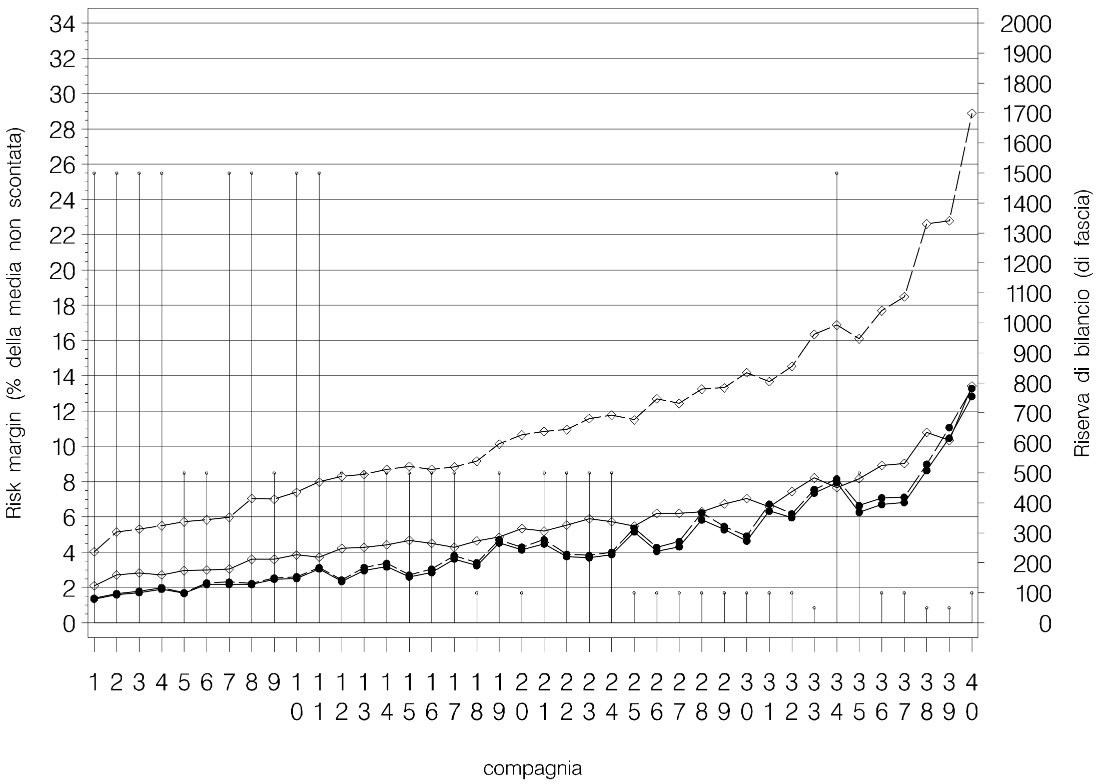

Figure 4.9: ODP model with con discounting: risk margin from quantiles and as costs of capital

CodCv	Class	$\mathrm{err}(\mathbf{Q}^{(75)})$ (%)	$\mathrm{err}(\mathbf{Q}^{(90)})$ (%)	$\mathrm{err}(\mathbf{Q}^{(95)})$ (%)	φ
1	1	0.08	0.15	0.20	0.9207
2	1	0.11	0.24	0.23	0.9271
3	1	0.11	0.24	0.23	0.9206
4	1	0.09	0.25	0.32	0.9177
5	2	0.12	0.22	0.25	0.9352
6	2	0.13	0.33	0.38	0.9127
7	1	0.08	0.18	0.25	0.9179
8	1	0.10	0.23	0.31	0.9342
9	2	0.18	0.29	0.40	0.9179
10	1	0.09	0.28	0.35	0.9193
11	1	0.12	0.36	0.51	0.9187
12	2	0.15	0.25	0.36	0.9322
13	2	0.22	0.50	0.62	0.9172
14	2	0.25	0.31	0.49	0.9149
15	2	0.22	0.27	0.37	0.9310
16	2	0.19	0.32	0.41	0.9208
17	2	0.23	0.44	0.62	0.9069
18	3	0.22	0.35	0.46	0.9211
19	2	0.27	0.53	0.72	0.9094
20	3	0.26	0.52	0.70	0.9144
21	2	0.16	0.48	0.68	0.9109
22	2	0.10	0.38	0.56	0.9223
23	2	0.23	0.37	0.59	0.9301
24	2	0.28	0.47	0.69	0.9249
25	3	0.23	0.67	0.90	0.9111
26	3	0.17	0.44	0.76	0.9277
27	3	0.24	0.51	0.62	0.9269
28	3	0.24	0.54	0.72	0.9015
29	3	0.18	0.39	0.50	0.9231
30	3	0.14	0.30	0.39	0.9287
31	3	0.21	0.52	0.55	0.9147
32	3	0.52	1.02	1.30	0.9182
33	4	0.16	0.48	0.49	0.9127
34	1	0.31	0.77	0.87	0.9097
35	2	0.18	0.49	0.69	0.9229
36	3	0.19	0.56	0.64	0.9235
37	3	0.39	0.64	0.84	0.9199
38	4	0.37	0.77	1.11	0.9236
39	4	0.58	1.50	2.10	0.9177
40	3	0.28	0.69	0.82	0.9174

Table 4.20: Percentage errors between approximated and exact discounting

CodCv	Class	K_0 ($\%M_0$)	K'_0 ($\%M_0$)	κ_0 ($\%K_0$)	κ_0 ($\%M_0$)	κ'_0 ($\%M_0$)
1	1	9.23	9.58	15.80	1.46	1.51
2	1	11.78	12.22	14.59	1.72	1.78
3	1	11.85	12.34	15.74	1.87	1.94
4	1	12.81	13.32	16.27	2.08	2.17
5	2	13.43	13.67	13.30	1.79	1.82
6	2	14.13	14.66	16.89	2.39	2.48
7	1	14.59	15.40	16.33	2.38	2.52
8	1	17.29	17.71	13.44	2.33	2.38
9	2	16.61	17.07	16.21	2.69	2.77
10	1	17.15	17.79	15.97	2.74	2.84
11	1	20.88	21.33	15.95	3.33	3.40
12	2	18.05	18.74	13.91	2.51	2.61
13	2	19.77	20.94	16.32	3.23	3.42
14	2	20.73	21.99	16.75	3.47	3.68
15	2	19.90	20.68	14.06	2.80	2.91
16	2	19.56	20.81	15.86	3.10	3.30
17	2	22.17	23.33	18.01	3.99	4.20
18	3	22.50	23.56	15.67	3.53	3.69
19	2	28.27	29.38	17.62	4.98	5.18
20	3	26.93	27.86	16.83	4.53	4.69
21	2	28.13	29.60	17.48	4.92	5.17
22	2	26.41	27.27	15.43	4.08	4.21
23	2	27.62	28.60	14.37	3.97	4.11
24	2	27.96	28.85	14.97	4.19	4.32
25	3	32.77	34.14	17.27	5.66	5.90
26	3	29.97	31.65	14.56	4.36	4.61
27	3	31.49	33.59	14.78	4.65	4.96
28	3	34.20	36.49	18.93	6.47	6.91
29	3	37.12	38.41	15.40	5.71	5.91
30	3	34.59	36.62	14.47	5.01	5.30
31	3	41.51	44.07	16.68	6.93	7.35
32	3	40.28	41.74	16.13	6.50	6.73
33	4	47.33	48.59	17.03	8.06	8.28
34	1	50.15	51.46	17.43	8.74	8.97
35	2	44.63	47.27	15.23	6.80	7.20
36	3	47.75	50.41	15.22	7.26	7.67
37	3	46.83	48.94	15.82	7.41	7.74
38	4	60.52	62.96	15.45	9.35	9.73
39	4	72.43	76.58	15.75	11.41	12.06
40	3	85.01	87.97	16.46	14.00	14.48

Table 4.21: ODP model with con discounting: risk capitals and costs of capital

4.3 Analysis of MTPL data by classes of companies

The analysis of market data has been also performed on an aggregate basis, considering the total paid losses for each dimensional class, instead of the individual companies. In this part of the study we considered all the companies present on the MTPL market at the end of 2004, totalling 75 companies; the distribution of the companies in the four classes just defined is illustrated in table 4.22.

Class	Statutory reserve	n. co.	Total reserve
1	$1,000 \leq R^s$	9	15,114.43
2	$250 \leq R^s < 1,000$	16	7,545.01
3	$50 \leq R^s < 250$	24	2,702.24
4	$0 \leq R^s < 50$	26	355.55
MTPL		75	25,717.22

Table 4.22: Classification of MTPL market by statutory reserve

For each class a single triangle of paid losses has been constructed summing the payments in the same (i, j) cell. The overall triangle – the "MTPL market triangle" – has been also derived by the same procedure. We applied the ODP model and the Mack's model to these aggregated triangles. The corresponding summary statistics of the predictive distribution of $\widehat{L}$ provided by the two models are reported in tables 4.23 and 4.24, that are analogous to table 4.1 and 4.2, respectively.

Class	R^s	$\overline{L}$	$\mathbf{Pstd}(L)$	$\mathbf{Cv}(L)$	$\mathbf{Q}^{(50)}(L)$	$\mathbf{Q}^{(75)}(L)$	$\mathbf{Q}^{(90)}(L)$	$\mathbf{Q}^{(95)}(L)$
1	15114.43	14378.68	489.849	3.41	14362.74	14692.44	15010.45	15208.82
2	7545.01	7662.27	255.683	3.34	7650.55	7827.61	7976.32	8064.85
3	2702.24	3040.72	155.932	5.13	3037.21	3140.67	3245.44	3315.90
4	355.55	510.39	109.821	21.52	512.94	583.19	657.51	701.98
MTPL	25717.22	25229.65	726.553	2.88	25217.71	25688.43	26142.00	26456.77

Table 4.23: ODP model – Summary statistics of L for different classes

Class	R^s	$\overline{L}$	$\mathbf{Pstd}(L)$	$\mathbf{Cv}(L)$	$\mathbf{Q}^{(50)}(L)$	$\mathbf{Q}^{(75)}(L)$	$\mathbf{Q}^{(90)}(L)$	$\mathbf{Q}^{(95)}(L)$
1	15114.43	14378.68	396.939	2.76	14373.20	14643.29	14890.71	15040.77
2	7545.01	7662.27	181.940	2.37	7660.12	7783.77	7896.76	7965.17
3	2702.24	3040.72	114.210	3.76	3038.58	3116.51	3188.36	3232.15
4	355.55	510.39	89.752	17.58	502.68	565.47	628.66	669.81
MTPL	25717.22	25229.65	623.441	2.47	25221.95	25645.79	26033.34	26268.07

Table 4.24: DFCL model – Summary statistics of L for different classes (lognormal assumption)

It should be pointed out that the variability measures in tables 4.23 and 4.24 are largely influenced by the diversification effect produced by the aggregation of the loss triangles. For example, the average of $\mathbf{Cv}(L)$ over the 40 companies in the selected sample resulted 9.99% for the ODP model (see table 4.7) and 7.27% for the Mack's model (table 4.8), which are much greater than the values 2.88% and 2.47% derived by the corresponding aggregate triangles. If the average is computed weighting by $\overline{L}$ this effect is only weakly reduced (one obtains $\mathbf{Cv}(L) = 7.17\%$ for the ODP and $\mathbf{Cv}(L) = 5.31\%$ for the DFCL).

In order to obtain a "representative loss distribution" of the overall MTPL market, we considered a lognormal distribution with mean equal to the total expected OLL provided by the chain-ladder and coefficient of variation equal to the average $\mathbf{Cv}(L)$ in the selected sample (hence 9.99% and 7.27%). These density functions for the two models are illustrated in figures 4.10 and 4.11. OLL values are reported in billions Euro. Also the sample mean $\overline{L}$ and the quantiles $\mathbf{Q}^{(\alpha)}(L)$, for $\alpha = 50, 75, 90, 95, 99, 99.5\%$ are indicated.

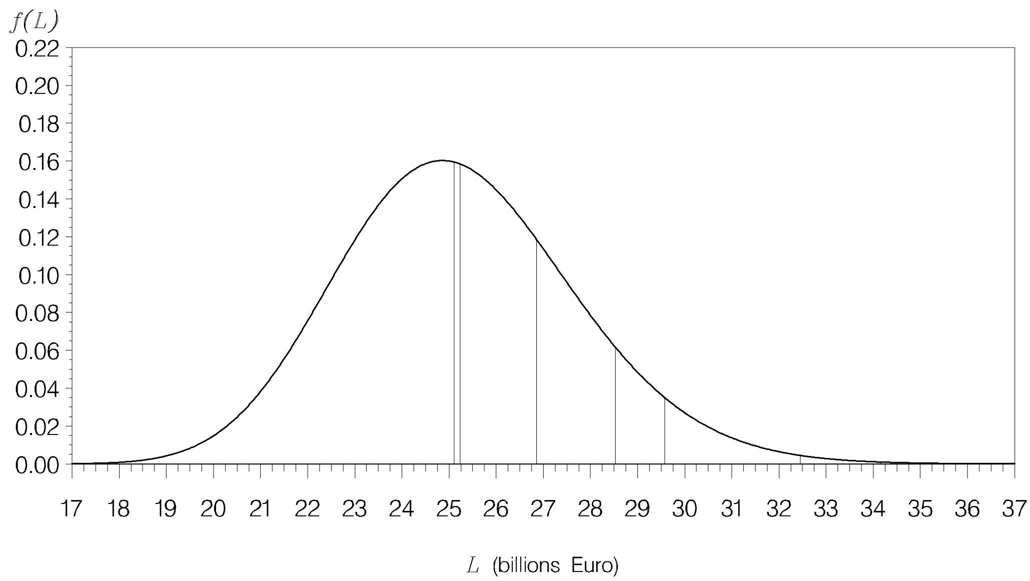

Figure 4.10: ODP model – Distribution of the MTPL overall OLL of all companies

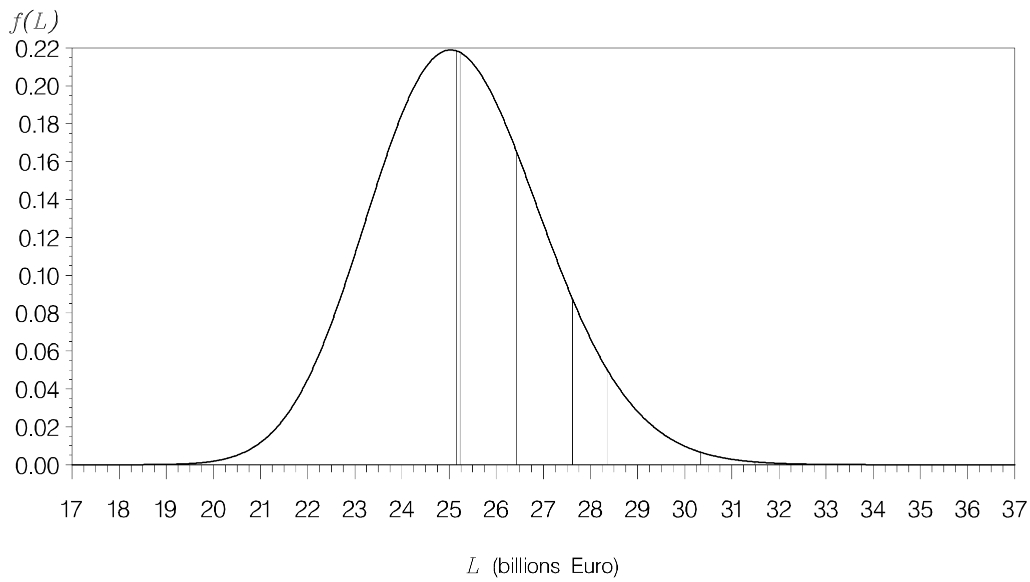

Figure 4.11: DFCL model – Distribution of the MTPL overall OLL of all companies

4.4 Analysis of Motor Kasko data

Data on the Motor Kasko (MK) have also been analyzed. For this line of business only aggregated loss triangles were available obtained summing the MK claim payments of companies in the same classes defined for MTPL in table 4.22. On December 31, 2004 seven years of data were available. The summary statistics of $\widehat{L}$ obtained by the ODP model and by the Mack's model applied to these triangles and on the overall triangle are reported in tables 4.25 and 4.26.

Class	R^s	$\overline{L}$	$\mathbf{Pstd}(L)$	$\mathbf{Cv}(L)$	$\mathbf{Q}^{(50)}(L)$	$\mathbf{Q}^{(75)}(L)$	$\mathbf{Q}^{(90)}(L)$	$\mathbf{Q}^{(95)}(L)$
1	350.12	325.74	21.40	6.57	325.24	339.79	353.61	362.56
2	189.28	157.98	7.88	4.99	157.84	163.27	168.10	171.03
3	86.93	79.00	6.60	8.35	79.11	83.69	87.80	90.27
4	64.33	38.68	7.85	20.29	38.45	43.89	49.17	52.31
MK	690.66	602.20	25.97	4.31	601.63	619.53	636.46	645.52

Table 4.25: MK data – Summary statistics of $\widehat{L}$ for different classes under ODP model

Class	R^s	$\overline{L}$	$\mathbf{Pstd}(\widehat{L})$	$\mathbf{Cv}(\widehat{L})$	$\mathbf{Q}^{(50)}(\widehat{L})$	$\mathbf{Q}^{(75)}(\widehat{L})$	$\mathbf{Q}^{(90)}(\widehat{L})$	$\mathbf{Q}^{(95)}(\widehat{L})$
1	350.12	325.74	15.170	4.66	325.39	335.76	345.39	351.28
2	189.28	157.98	9.786	6.19	157.68	164.40	170.69	174.58
3	86.93	79.00	7.263	9.19	78.67	83.69	88.49	91.49
4	64.33	38.68	9.720	25.13	37.52	44.33	51.51	56.36
MK	690.66	602.20	22.565	3.75	601.77	617.17	631.37	640.02

Table 4.26: MK data – Summary statistics of $\widehat{L}$ for different classes under DFCL model (lognormal assumption)

4.5 Comparing results from multiple approaches

In chapter 1 we considered a number of alternative methods and approximations for computing risk margins and risk capitals. The formal representation of these approaches has been summarized in tabular form in section 1.3.3. The outcomes of the empirical analysis presented in some details in the previous part of this chapter however, were referred to only few of these methods. In this section we provide in synthetic form the results obtained by applying to our MTPL data all of the methods and approximations introduced in chapter 1; the exposition strictly follows [16][1]. We computed all the items considered in tables 1.1, 1.2 and 1.3 applying the ODP and the DFCL model to the triangle of historical paid losses of each of the 40 companies in the selected sample. The results of the valuation procedures are summarized in tables 4.27, 4.28 and 4.29, which are arranged exactly as the definitory tables 1.1, 1.2 and 1.3, respectively. All figures are expressed as a percent of the corresponding undiscounted best estimate (BE) $\overline{L}$ (the chain-ladder estimate of the OLL). For each item the sample mean, the standard deviation, the minimum and the maximum value in the sample are reported[2].

As a first general result it turns out that the DFCL model produces in the sample variability estimates uniformly lower than that provided by the ODP model; this is consistent with the average value of the coefficient of variation of L obtained with the two models, which is 6.81% for the Mack model and 9.03% for the ODP model.

Risk margins and required reserves. In table 4.27 numerical results corresponding to table 1.1 are provided. The required reserve (RR) is computed as the 75-th and the 90-th quantile of the relevant random variable (RRV) and the corresponding risk margin (RM) is derived. One can observe that the 75-th quantile risk margins for the ODP are about 35÷40% higher than the corresponding risk margins for the DFCL. In both models risk margin figures display large variability across the sample. For both models the reference level 100 of the undiscounted best estimate is greater than the minimum value and lower than the maximum value of the required reserve computed as the discounted 75-th quantile.

Reserve risk capitals given risk margins. In table 4.28 we summarize results of risk capital (RC) computations after risk margins have been derived under an α-quantile assumption, with $\alpha = 75\%$ and 90%. The RAV has been defined as the quantile of the RRV (see table 1.2) at 99.5% security level. The first year risk loading has been computed as $\gamma_1^{(\alpha)} = \beta_1 \, \mathbf{Q}_0^{(\alpha)}(D) - \overline{Y}_1$, where the allocation fraction has been fixed following (1.50) as $\beta_1 = \overline{Y}_1 / M_0$. As for the risk margins, also the risk capital values display high variability across the sample.

[1] Risk capital values from quantiles considered in [16] (and in previous versions of this paper) were computed at the 99.9% confidence level (as in the "Basel 2" framework). Here we refer instead to a 99.5% confidence level, as recently prescribed from CEIOPS [9]. In addition we also report here the valuations obtained with the discounted approach under the DFCL model; these figures have been derived using the closed form expressions for the correlations between the Y_τ "diagonals" provided in [34].

[2] To allow a better comparison the results of the ODP model have been adjusted for the simulation error as illustrated in chapter 4: from all figures the difference $(\widetilde{L} - \overline{L})$ has been subtracted, where $\widetilde{L}$ is the sample mean of the total OLL provided by the simulation.

		ODP model				Mack model			
		mean	stdd	min	max	mean	stdd	min	max
disc	BE	92.0	0.8	90.2	93.5	92.0	0.8	90.2	93.5
	RR75	97.7	2.5	94.2	105.2	96.3	2.1	93.4	102.6
	RM75	5.7	2.5	2.1	13.4	4.3	2.0	1.6	10.6
	RR90	103.5	5.4	96.1	120.6	100.6	4.4	95.2	115.0
	RM90	11.5	5.4	4.0	28.9	8.7	4.3	3.1	23.1
φ-disc	BE	92.0	0.8	90.2	93.5	92.0	0.8	90.2	93.5
	RR75	97.9	2.6	94.3	105.5	96.3	2.2	93.4	103.1
	RM75	5.9	2.6	2.2	13.7	4.3	2.0	1.6	11.1
	RR90	104.0	5.6	96.2	121.5	100.7	4.5	95.1	116.2
	RM90	12.0	5.6	4.2	29.7	8.7	4.4	3.1	24.2
undisc	BE	100.0	0.0	100.0	100.0	100.0	0.0	100.0	100.0
	RR75	106.4	2.8	102.4	115.0	104.7	2.2	101.7	112.0
	RM75	6.4	2.8	2.4	15.0	4.7	2.2	1.7	12.0
	RR90	113.0	6.1	104.5	132.4	109.5	4.8	103.3	126.3
	RM90	13.0	6.1	4.5	32.4	9.5	4.8	3.3	26.3

Table 4.27: Risk margins where the required reserve is defined as 75-th and 90-th quantile of the OLL.

		ODP model								Mack model			
		YEE approach				LM approach				LM approach			
		mean	stdd	min	max	mean	stdd	min	max	mean	stdd	min	max
disc	RAV	121.3	16.2	101.1	174.3	117.1	12.6	100.2	155.2	110.8	10.3	98.4	146.8
	BE	92.0	0.8	90.2	93.5	92.0	0.8	90.2	93.5	92.0	0.8	90.2	93.5
	RC75	27.0	15.4	8.2	78.2	22.8	11.8	7.4	59.1	17.1	9.4	5.7	50.2
	RC90	24.7	14.3	7.4	73.2	20.5	10.6	6.6	54.1	15.3	8.4	5.2	44.9
φ-disc	RAV	123.1	17.2	101.5	176.9	118.3	13.3	100.6	157.2	111.0	10.7	98.4	150.0
	BE	92.0	0.8	90.2	93.5	92.0	0.8	90.2	93.5	92.0	0.8	90.2	93.5
	RC75	28.9	16.4	8.6	80.8	24.0	12.5	7.7	61.1	17.2	9.8	5.7	53.3
	RC90	26.5	15.2	7.9	75.8	21.7	11.4	7.0	56.1	15.5	8.7	5.1	47.6
undisc	RAV	133.9	18.9	110.3	192.9	128.6	14.7	109.3	171.4	120.6	11.5	106.9	163.1
	BE	100.0	0.0	100.0	100.0	100.0	0.0	100.0	100.0	100.0	0.0	100.0	100.0
	RC75	31.6	17.9	9.4	88.4	26.3	13.7	8.4	66.9	18.8	10.7	6.2	58.2
	RC90	29.2	16.8	8.7	83.3	23.9	12.6	7.7	61.8	17.0	9.6	5.6	52.4
flat	RAV	133.9	18.9	110.3	192.9	128.6	14.7	109.3	171.4	120.6	11.5	106.9	163.1
	BE	100.0	0.0	100.0	100.0	100.0	0.0	100.0	100.0	100.0	0.0	100.0	100.0
	RC	33.9	18.9	10.3	92.9	28.6	14.7	9.3	71.4	20.6	11.5	6.9	63.1

Table 4.28: Risk capitals derived by 99.5-th quantiles given risk margins derived by 75-th and 90-th quantiles.

Risk capitals and risk margins as the costs of capital. Table 4.29 illustrates the outcomes of risk margin computations defined as the cost of the reserve risk capital, consistently with the formulas summarized in table 1.3. Also in this case the RAV has been defined as the 99.5-th quantile of the RRV and the risk premium to the shareholders has been fixed at 6%. The results confirm that under these assumptions the risk margins derived as the cost of capital with the YEE approach are not very different, but typically lower than the corresponding risk margins as 75-th quantile provided in table 4.27. For example, the average RM reported in table 4.29 for the discounted case is 4.4, while the average for RM75 in table 4.27 is 5.7. The minimum and the maximum value in the sample for the cost-of-capital approach

		ODP model								Mack model			
		YEE approach				LM approach				LM approach			
		mean	stdd	min	max	mean	stdd	min	max	mean	stdd	min	max
disc	RAV	121.3	16.2	101.1	174.3	117.1	12.6	100.2	155.2	110.8	10.3	98.4	146.8
	BE	92.0	0.8	90.2	93.5	92.0	0.8	90.2	93.5	92.0	0.8	90.2	93.5
	RC	27.7	15.4	8.5	78.0	23.7	12.0	7.7	60.0	17.8	9.7	6.0	51.8
	RM	4.4	2.5	1.3	12.8	3.8	2.0	1.2	9.9	2.8	1.5	1.0	8.2
φ-disc	RAV	123.1	17.2	101.5	176.9	118.3	13.3	100.6	157.2	111.0	10.7	98.4	150.0
	BE	92.0	0.8	90.2	93.5	92.0	0.8	90.2	93.5	92.0	0.8	90.2	93.5
	RC	29.4	16.3	8.9	80.5	24.8	12.7	8.0	61.8	17.9	10.0	6.0	54.8
	RM	4.7	2.7	1.4	13.3	4.0	2.1	1.3	10.2	2.8	1.6	0.9	8.7
undisc	RAV	133.9	18.9	110.3	192.9	128.6	14.7	109.3	171.4	120.6	11.5	106.9	163.1
	BE	100.0	0.0	100.0	100.0	100.0	0.0	100.0	100.0	100.0	0.0	100.0	100.0
	RC	32.0	17.8	9.7	87.6	27.0	13.8	8.7	67.3	19.5	10.9	6.5	59.5
	RM	5.6	3.2	1.7	15.7	4.7	2.5	1.5	12.1	3.4	1.9	1.1	10.3
flat	RAV	133.9	18.9	110.3	192.9	128.6	14.7	109.3	171.4	120.6	11.5	106.9	163.1
	BE	100.0	0.0	100.0	100.0	100.0	0.0	100.0	100.0	100.0	0.0	100.0	100.0
	RC	33.9	18.9	10.3	92.9	28.6	14.7	9.3	71.4	20.6	11.5	6.9	63.1
	RM	5.9	3.4	1.8	16.6	5.0	2.6	1.6	12.8	3.6	2.0	1.2	10.9

Table 4.29: Risk margins as the cost of risk capital derived by 99.5-th quantiles (spread $s = 6\%$).

is 1.3 and 12.8, respectively; the corresponding values for the quantile approach are for 2.1 and 13.4. The standard deviation is equal in the two cases. An analogous dominance relation holds comparing corresponding figures under both the φ-discounted and the undiscounted approach.

The table also confirms the general finding that for a given approximation (discounted, φ-discounted, undiscounted and flat) risk measures provided by ODP with the YEE approach are typically higher than risk measures obtained under ODP with the LM approach; which in turn are higher than that produced by the Mack model with the corresponding approximation.

These results could be of some concern for fairly implementing a market regulation based on the use of internal models.

4.6 A comparison with the QIS2 capital requirements

It could be interesting to compare the risk capital figures provided by the stochastic models with the solvency capital requirement (SCR) for reserve risk computed as prescribed by CEIOPS in the Solvency 2 framework. In the technical specifications for CEIOPS' second Quantitative Impact Study (QIS2) issued in May [9], a factor-based approach is proposed where the SCR for reserve risk $K^{(QIS)}$ is derived following the following steps.

a) 11 lines of business (LoB) for non-life insurance are defined; for each LoB k a market-wide *volatility factor* f_k is specified.

b) The LoB volatility for each company is computed as:

$$\sigma_k = f_k \, s(\widetilde{R}_k) \, ,$$

where $\widetilde{R}_k$ is the "provision for claims outstanding" (gross of reinsurance) of the LoB, and the *size factor* s is a non-increasing function of $\widetilde{R}_k$, specified as:

$$s(\widetilde{R}_k) = \begin{cases} 1 & if\ \widetilde{R}_k \geq 100, \\ \frac{10}{\sqrt{\widetilde{R}_k}} & if\ 100 > \widetilde{R}_k \geq 20, \\ \frac{10}{\sqrt{20}} & if\ \widetilde{R}_k < 20, \end{cases}$$

where $\widetilde{R}_k$ is expressed in million Euros.

c) The volatility (for unit of provision) σ relative to the non-life business is computed aggregating the individual volatilities as:

$$\sigma^2 = \sum_{k=1}^{11} \sum_{j=1}^{11} w_k\, w_j\, \sigma_k\, \sigma_j\, c_{kj}\ ,$$

where $\{c_{kj}\}$ is a specified correlation matrix across LoBs, and the weights are defined as:

$$w_k := \frac{R_k}{\sum_{j=1}^{11} R_j}\ ,$$

R_k being the net provision for claims outstanding of LoB k.

d) The "Basic Solvency Capital Requirement" BSCR for the reserve risk is derived as:

$$\mathsf{BSCR} = \rho(\sigma) \sum_{k=1}^{11} R_k\ ,$$

where the ρ function has the form:

$$\rho(x) := \frac{0.99 - N\Big(N^{-1}(0.99) - \sqrt{\log(x^2+1)}\Big)}{0.01}\ , \tag{4.2}$$

$N(x)$ being the cumulative distribution function of the standard normal variate.

e) The SCR is finally obtained as:

$$K^{(\mathsf{QIS})} := \mathsf{BSCR} - \mathsf{PL}\,,$$

where PL represents the "expected profit or loss arising from next year's business". As specified in the technical document (pp. 20-22), PL is given by the sum of a component $\mathsf{PL}_{\mathsf{prem}}$ relative to premiums and a component $\mathsf{PL}_{\mathsf{res}}$ relative to reserves. The $\mathsf{PL}_{\mathsf{prem}}$ component expresses the expectation of the difference between the net earned premiums and the corresponding costs (claims payments plus expenses) in the forthcoming year. The quantity $\mathsf{PL}_{\mathsf{res}}$ is essentially given by the fraction of the risk margin allocated in the year $\tau = 1$ (as a proportion of the claims provisions). The two components are detracted at the overall business level; however they are defined and computed for each of the LoBs.

In the QIS2 framework, MTPL insurance is the LoB $k = 2$ and the corresponding market-wide volatility factor is $f_2 = 0.15$. Since we are only concerned here with the MTPL, the SCR computations have been performed at a single-LoB level, skipping step (c) (or posing

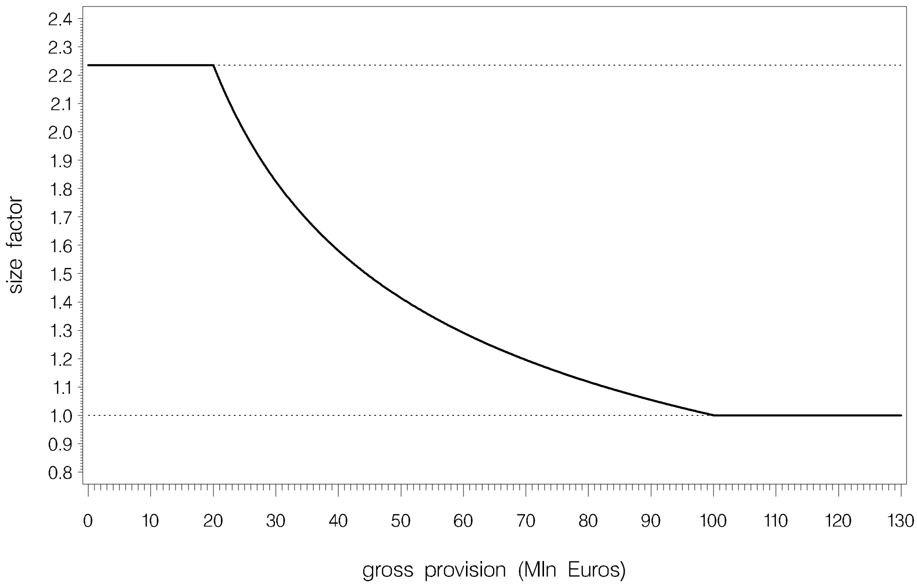

Figure 4.12: The size factor function

$R_k = 0$ for $k \neq 2$) and considering $K^{(QIS)}$ for MTPL as a stand-alone risk capital. Thus also the expected profit or loss PL have been computed for the single LoB.

The volatility factor f_2 must be multiplied by the company-specific size factor; the form of this function is illustrated in figure 4.12.

It can be easily shown (see for ex. [6], pp. 383) that the expression (4.2) of the $\rho(x)$ function is the TailVaR at the 99% confidence level of a lognormal random variable with mean 1 and standard deviation x. In the technical document a broad assumption is explicitly made that TailVaR 99% is "an equivalent level of prudence" of VaR 99.5% (p. 4). Thus risk capital figures provided by stochastic models can be consistently compared with the SCR prescribed by CEIOPS defining the RAV as the quantile at the level $\alpha = 99.5\%$.

Following the QIS2 prescriptions PCO has been computed as the 75-th quantile of the discounted OLL; that is:

$$R_2 := \mathbf{Q}^{(75)}(D)\,;$$

the percentile has been taken on probability distribution produced by the ODP model. The PL_{res} component of the expected profit or loss has been obtained as[3]:

$$PL_{res} := \left[\mathbf{Q}^{(75)}(D) - M_0\right] \frac{\overline{Y}_1}{\overline{L}}\,.$$

We did not considered the PL_{prem} component; reinsurance effects have been also ignored assuming $\widetilde{R}_2 = R_2$.

In table 4.30 we reported the MTPL reserve risk capital of all the companies in the selected sample, computed following QIS2 prescriptions and applying all the 12 methods and approximations previously described: discounted, φ-discounted, undiscounted and flat

[3]This quantity turns out to be very similar to the first year risk loading γ_1 defined in section 1.2 and specified by the allocation rule (1.50).

	QIS2	**ODP model**								**Mack model**			
CodCv		**YEE approach**				**LM approach**				**LM approach**			
		disc	φ-dsc	undsc	flat	disc	φ-dsc	undsc	flat	disc	φ-dsc	undsc	flat
1	43.81	8.49	8.93	9.69	10.27	7.72	8.05	8.73	9.26	6.02	5.97	6.47	6.86
2	44.00	10.92	11.78	12.69	13.45	9.99	10.32	11.12	11.79	9.27	9.42	10.14	10.75
3	43.82	10.91	11.67	12.66	13.42	10.24	10.67	11.57	12.27	10.19	10.00	10.84	11.49
4	44.28	12.56	13.26	14.16	15.01	10.98	11.22	11.98	12.70	7.99	8.02	8.56	9.08
5	43.75	11.75	12.42	13.52	14.33	10.87	11.28	12.28	13.02	8.70	8.56	9.32	9.88
6	43.49	12.89	13.66	14.94	15.84	10.93	11.53	12.62	13.37	8.07	7.98	8.73	9.25
7	43.86	13.39	14.09	15.34	16.26	12.11	12.38	13.47	14.28	9.40	9.31	10.13	10.74
8	43.89	15.76	16.46	17.89	18.96	14.23	14.62	15.89	16.84	11.25	11.17	12.13	12.86
9	44.24	16.16	17.10	18.28	19.38	14.26	14.89	15.92	16.87	11.30	11.25	12.02	12.74
10	43.82	15.25	16.20	17.62	18.68	14.50	15.11	16.45	17.43	11.43	11.68	12.71	13.47
11	44.31	16.83	17.63	18.89	20.03	16.04	16.55	17.73	18.80	13.52	13.58	14.55	15.42
12	44.28	18.53	19.61	21.04	22.30	16.11	16.74	17.96	19.04	13.95	13.97	14.99	15.89
13	43.87	18.14	20.12	21.91	23.23	16.01	17.06	18.58	19.69	13.61	13.50	14.71	15.59
14	43.89	18.96	20.17	22.02	23.34	16.82	17.69	19.31	20.47	11.34	11.27	12.30	13.04
15	43.80	19.18	20.10	21.85	23.16	17.21	18.09	19.67	20.85	10.32	10.21	11.10	11.76
16	44.11	18.01	19.09	20.70	21.94	17.26	18.29	19.83	21.02	14.45	14.39	15.60	16.54
17	43.55	20.11	21.45	23.62	25.04	17.26	18.14	19.98	21.18	19.41	20.54	22.62	23.97
18	44.02	20.72	22.26	24.13	25.58	18.67	19.81	21.48	22.77	12.63	12.76	13.84	14.67
19	43.98	24.63	26.26	28.68	30.40	20.62	21.83	23.85	25.28	13.10	13.12	14.33	15.19
20	43.76	25.71	27.30	29.98	31.78	21.00	22.33	24.52	25.99	13.11	13.11	14.40	15.27
21	44.52	25.69	26.88	28.86	30.59	22.20	22.84	24.52	25.99	17.73	17.79	19.11	20.25
22	44.11	24.36	25.95	28.09	29.78	22.06	23.13	25.05	26.55	14.92	14.87	16.10	17.07
23	43.95	25.62	27.44	30.09	31.89	22.79	24.06	26.38	27.96	13.91	13.83	15.17	16.08
24	44.12	25.86	27.77	29.99	31.79	23.57	24.58	26.54	28.13	19.83	20.18	21.79	23.10
25	43.80	29.86	32.22	35.32	37.43	24.60	26.26	28.79	30.52	14.83	15.17	16.63	17.62
26	44.30	29.19	31.55	33.99	36.03	26.14	26.92	29.01	30.75	22.27	22.86	24.64	26.11
27	65.23	27.81	29.93	32.22	34.15	26.05	26.96	29.02	30.76	23.32	23.87	25.70	27.24
28	52.07	34.26	35.98	38.93	41.27	27.67	28.76	31.12	32.98	16.14	16.11	17.43	18.47
29	59.43	32.12	33.80	36.35	38.53	27.91	28.97	31.15	33.02	24.23	24.66	26.52	28.11
30	57.75	30.83	32.74	36.27	38.45	28.25	29.58	32.77	34.73	22.74	23.44	25.97	27.53
31	44.22	36.99	40.27	43.80	46.43	28.75	31.00	33.72	35.74	18.58	18.48	20.10	21.31
32	63.41	37.97	40.45	44.17	46.82	31.75	33.62	36.71	38.91	17.65	18.01	19.67	20.85
33	44.04	45.62	47.99	52.69	55.85	35.08	36.56	40.13	42.54	20.74	20.54	22.55	23.90
34	71.11	43.20	44.55	48.75	51.67	35.92	36.70	40.16	42.57	18.62	18.67	20.42	21.65
35	45.86	44.10	47.73	51.61	54.71	36.35	37.66	40.72	43.16	43.69	44.76	48.41	51.31
36	44.03	41.18	43.81	47.41	50.25	36.18	38.20	41.34	43.82	36.95	36.34	39.33	41.69
37	62.93	43.08	46.34	50.31	53.33	39.40	41.48	45.03	47.73	51.78	54.82	59.52	63.09
38	84.94	55.90	58.21	62.95	66.72	47.53	49.98	54.04	57.29	31.61	31.65	34.23	36.28
39	71.82	66.47	73.16	79.62	84.40	52.46	57.86	62.97	66.74	25.23	24.74	26.92	28.54
40	45.68	77.99	80.48	87.62	92.87	59.97	61.85	67.33	71.37	26.62	26.64	29.00	30.74

Table 4.30: Reserve risk capitals as a percent of the undiscounted BE. Values from QIS2 and from stochastic models under different approaches.

approach, with both the YEE and the LM method for the ODP model, and with LM method for the DFCL. The risk capitals under stochastic models are obtained assuming the RAV as the 99.5-th quantile of the RRV and the risk margins have been defined as the cost of capital (with the usual 6% spread); hence the formulas in table 1.3 have been used.

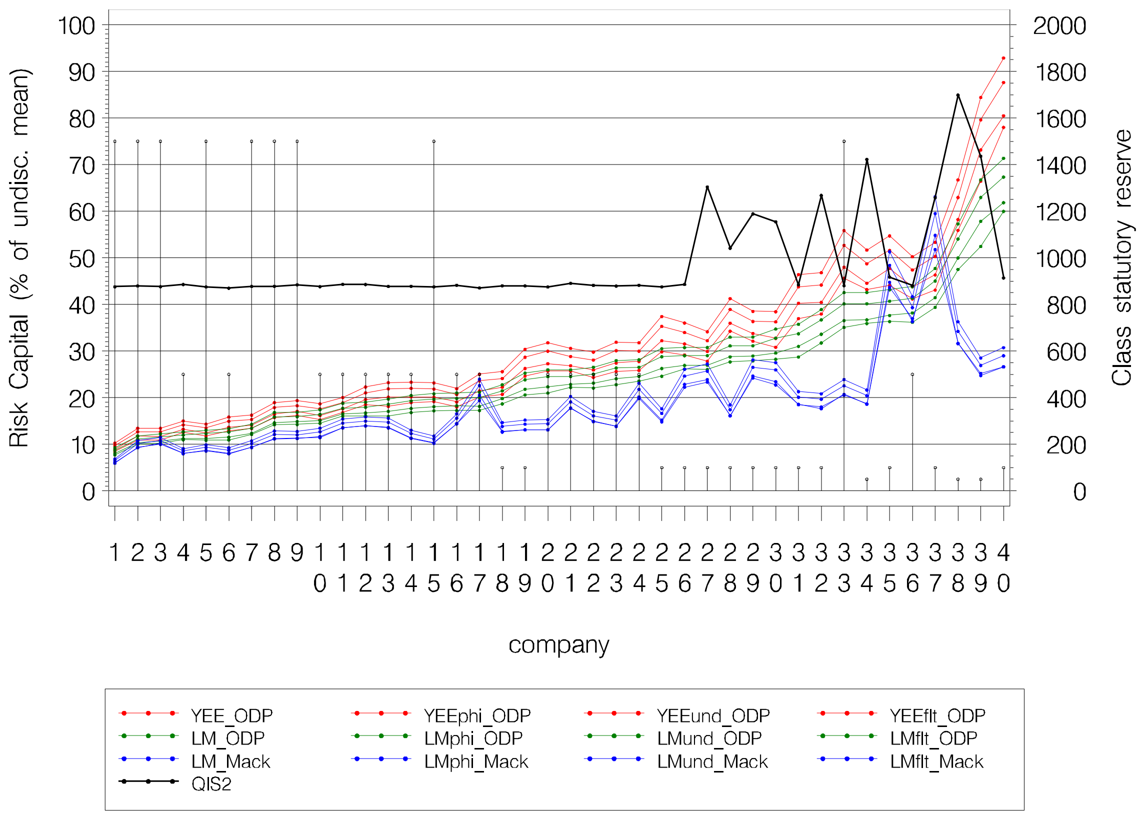

Figure 4.13: Reserve risk capitals following QIS2 and under stochastic models – QIS2: 99% expected shortfall; stochastic: derived by 99.5% quantiles (RM as CoC, $s = 6\%$)

As in the previous tables, the risk capitals from stochastic models are expressed as a percent of $\overline{L}$; the QIS2 risk capitals are expressed as a percent of the statutory reserve, which is the same as to report the values of the ρ function (in percent). Companies are sorted by CodCv, that is by increasing values of the $\mathbf{Cv}(L)$ provided by the ODP model. Data in table 4.30 are graphically illustrated in figure 4.13, where we also reported the usual representation of the conventional of statutory reserve of the dimensional classes (just defined as $1500, 500, 100$ and 50 for class $1, 2, 3$ and 4, respectively).

The SCR values are nearly equal for companies with reserve level larger than 100 million Euros (size factor $s(\widetilde{R}_k) = 1$), since the PL term, which is the only entity-specific parameter for these companies, is very low with respect to the BSCR. For almost all these companies an internal model based on any of the stochastic approaches considered should allow a sensible reduction of the capital requirement with respect to the QIS2 prescriptions.

Chapter 5

Applying the stochastic models to inflation-adjusted data

5.1 Adjusting for past inflation

Claim costs are typically subject to inflation. Usually the claims inflation can be considered as a composition of the *economic inflation* (the usual price or wage inflation) and a *superimposed inflation* which is an escalation of claim costs specific of the line of business under consideration. When the triangles of paid losses are expressed in terms of historical costs, the traditional run-off techniques for loss reserving implicitly assume the trend of past claims inflation as embedded into the cost development rule; therefore projected paid losses will include the inflation trend experienced in the past. The same effect will be observed under a stochastic loss reserving model, with the additional consequence that the variability of past inflation can influence also the variability of the predictive OLL distribution.

In order to control these inflactionary effects the run-off techniques (both deterministic and stochastic) can be applied in the following three steps:

- estimate a time series of claims inflation rates for the specific line of business and escalate the paid losses on the same diagonal of the past triangle by the corresponding inflation rate;
- apply the loss reserving model to the triangle of inflation-adjusted paid losses;
- escalate the projected paid losses under a model (deterministic or stochastic) for future inflation.

Under a stochastic approach the model for projected inflation should allow both for expected inflation and for inflation volatility and the last two steps should be integrated into an uncertainty model for both technical cost development and claims inflation.

5.1.1 Estimation of historical claims inflation

In order to derive a time series of past inflation for MTPL claim costs also data on claim counts have been considered. For each company the loss payments $C_{i,j}$ in the past triangle have been divided by the corresponding number $N_{i,j}$ of paid claims and a triangle of (*average*) *cost per claim paid*(CPCP):

$$\overline{C}_{i,j} := \frac{C_{i,j}}{N_{i,j}}\,, \quad i = 1, 2, \ldots, n\,, \quad j = 1, 2, \ldots, d_i\,,$$

has been obtained. For each DY $j = 1, 2, \ldots, n$ the set of CPCP:

$$\left\{\overline{C}_{i,j},\ i = 1, 2, \ldots, d_j\right\},$$

provides the time series of the average unitary costs experienced by the company in the accounting year i for claims paid with a delay j. On the chosen triangles, the time series $\left\{\overline{C}_{i,j}\right\}$ will contain $d_j = 10 - j + 1$ observations, from the accounting year $1994 + j$ to the accounting year 2004. In particular, for $j = 1$ the time series is obtained of ten CPCP values observed from 1995 ($i = 1$) to 2004 ($i = 10$) for claims paid in the first development year[1].

In figure 5.1 the time series $\left\{\overline{C}_{i,1},\ i = 1, 2, \ldots, 10\right\}$ of the CPCP experienced by the selected companies for paid losses with one development year are illustrated. The same figures for loss payments made in the second development year are reported in figure 5.2.

For each DY we also computed the average of the CPCP taken over all the companies (with equal weights). The time series of these average costs for DY from 1 to 9 are illustrated in figure 5.3. As expected, the CPCP dramatically increase with the payment delay. Moreover the time variability of CPCP is also strongly increasing with DY, reflecting the higher uncertainty affecting the claim cost of more delayed payments. It seems reasonable to conjecture that the information on the systematic component of the MTPL claims inflation is essentially contained in the time evolution of the CPCP for the first DY. Hence the estimation of claims inflation has been derived only by the time series $\left\{\overline{C}_{i,1}\right\}$. For each company the time series of annual rates of inflation from 1996 to 2004 have been derived by the corresponding ten-year time series of costs and the inflation rates in the same calendar year i have been averaged over all the companies using different weighting criteria. The time series of annual rates of inflation r_i obtained in this way are reported in table 5.1 where:

- $r_i^{(E)}$: equally weighted average rate;
- $r_i^{(R)}$: average rate weighted by statutory reserve R^s;
- $r_i^{(P)}$: average rate weighted by total paid losses S;
- $r_i^{(U)}$: average rate weighted by total ultimate loss U projected by the chain-ladder;
- $r_i^{(e)}$: economic inflation rate.

On the bottom lines of the table the average annual rates (geometric mean) of claims inflation for the overall time period 1996 to 2004 are also provided, together with the corresponding volatilities. The time series of the different inflation rates are illustrated in figure 5.4. In all the series the annual inflation rates are high and display high variability with respect to economic inflation; in each case the average annual rate of claims inflation on the overall period is greater than 9%.

It could be interesting to derive a rough estimate of the average cost per claims paid in different DY. Given the figures in table 5.1, a global inflation rate $r = 9\%$ could be used for escalating all the costs $\overline{C}_{i,j}$, providing:

$$\overline{C}^*_{i,j} := \overline{C}_{i,j}\,(1 + r)^{(i-1)}.$$

These amounts can be assumed as expressed in 2004 money values and can then be correctly

[1] The CPCP values were computed for only 39 companies in the selected sample, since count data contained an evident error for one company.

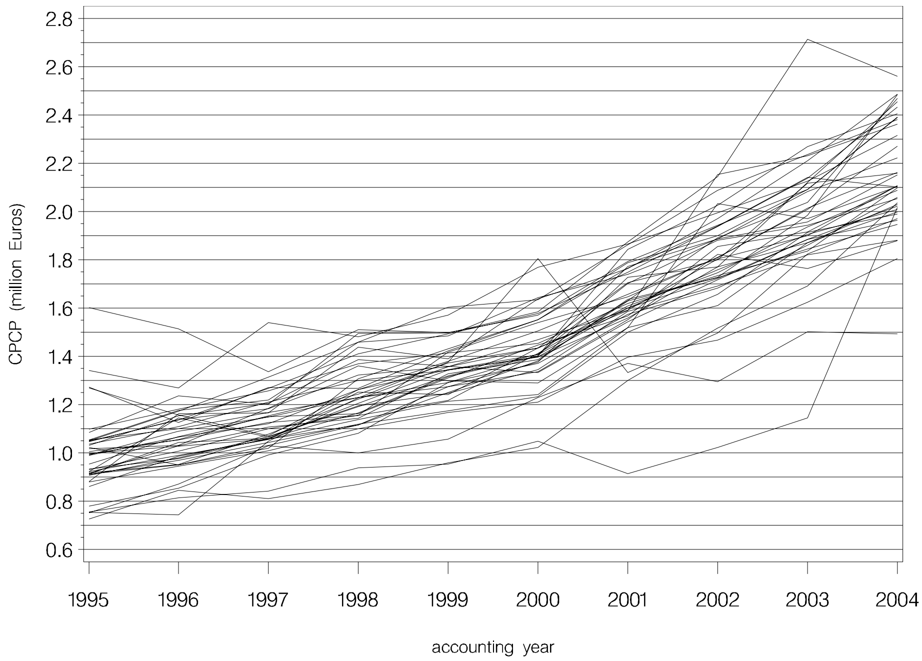

Figure 5.1: Costs per claims paid in the first development year

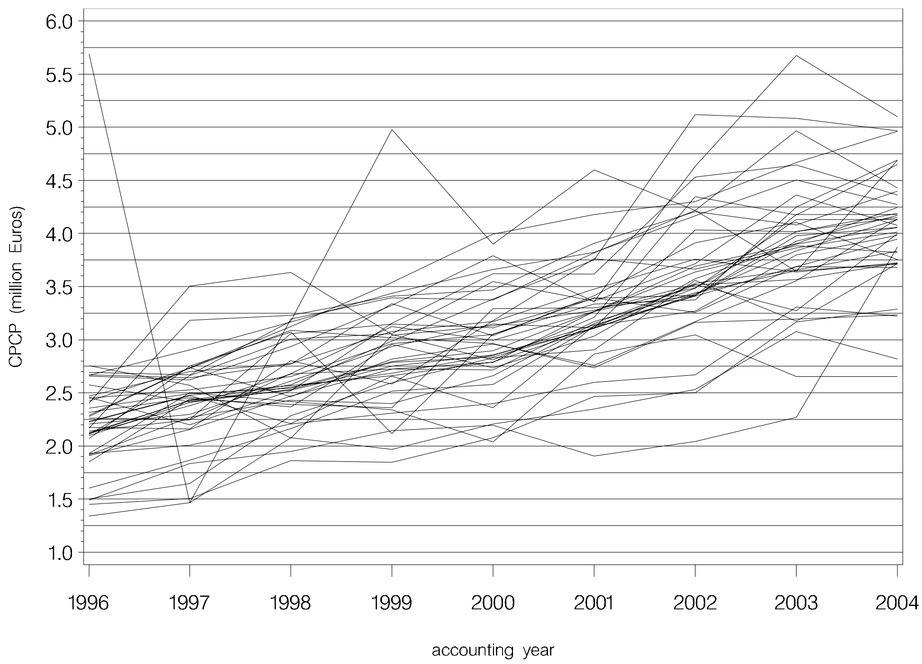

Figure 5.2: Costs per claims paid in the second development year

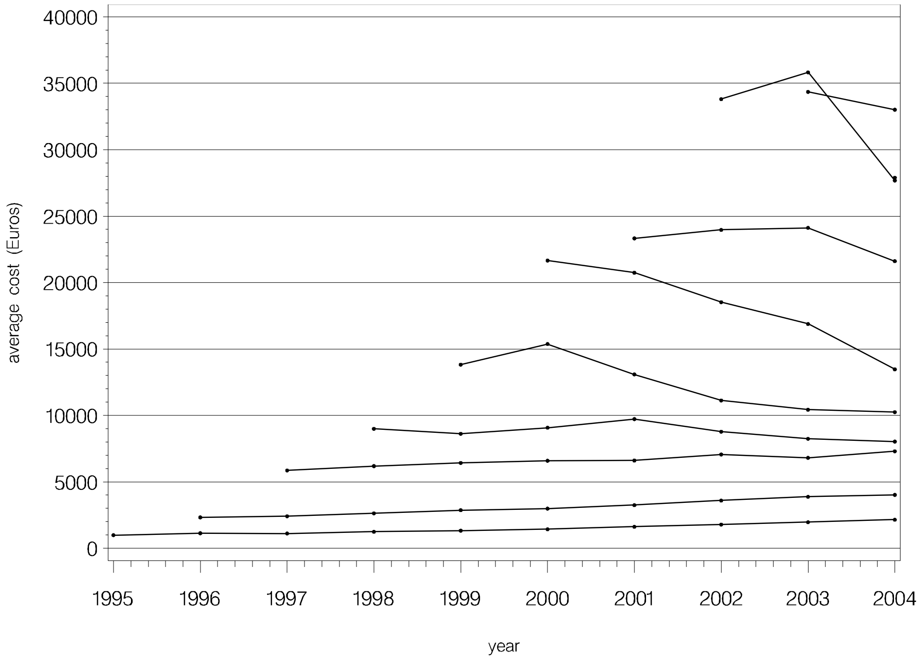

Figure 5.3: Average costs per claims paid in different DY

year	$r^{(E)}$	$r^{(R^s)}$	$r^{(P)}$	$r^{(U)}$	$r^{(e)}$
1996	7.66	5.99	5.70	5.76	3.90
1997	7.26	6.33	6.55	6.50	1.70
1998	10.25	11.60	11.60	11.60	1.80
1999	7.07	5.96	6.35	6.27	1.60
2000	7.84	6.68	6.80	6.77	2.60
2001	13.50	12.82	13.82	13.60	2.70
2002	10.49	9.29	9.19	9.21	2.40
2003	10.11	9.29	9.25	9.26	2.50
2004	10.23	14.99	13.26	13.65	2.00
average	9.36	9.17	9.13	9.14	2.35
volatility	1.91	3.01	2.83	2.86	0.69

Table 5.1: Inflation rates

averaged, for each DY, over all the accounting years from 1995 to 2004.

$$\overline{C}_j^* := \frac{1}{d_j} \sum_{i=1}^{d_j} \overline{C}_{i,j}^* \,.$$

The equally weighted average over all companies of the inflation adjusted CPCP $\overline{C}_j^*$ as a function of DY is illustrated in figure 5.5 with dots joined by solid line. The average CPCP plus and minus one standard deviation, computed on the selected company sample, are also reported with dotted lines.

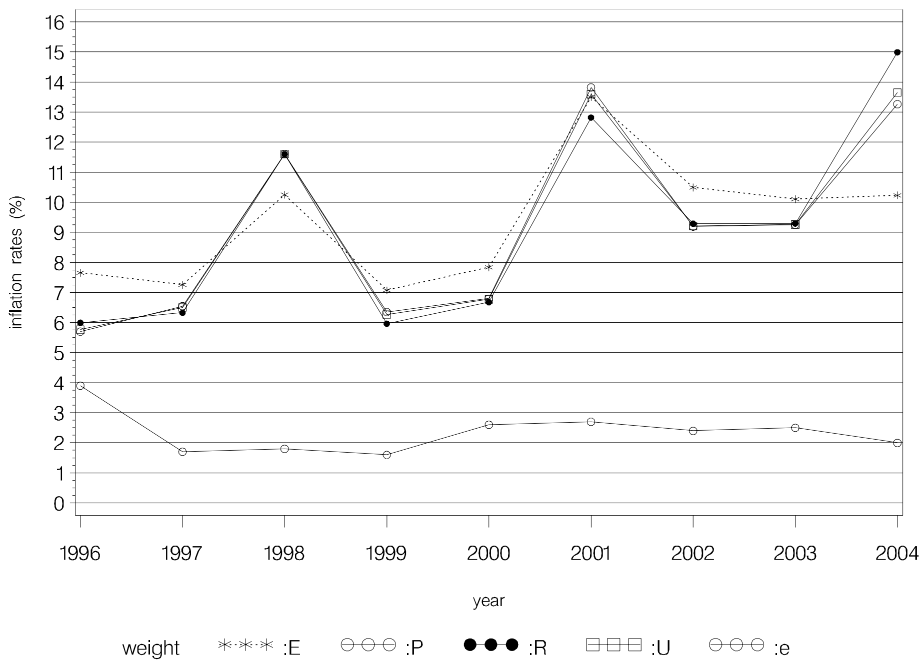

Figure 5.4: Inflation rates

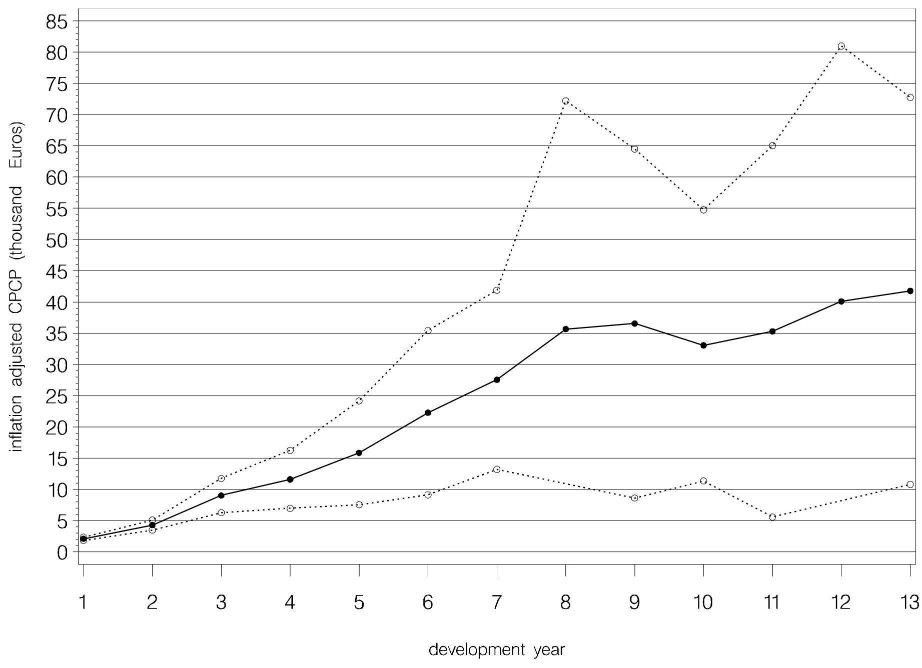

Figure 5.5: Inflation adjusted costs per claim paid in different DY

5.1.2 Adjusting for changes in speed of finalization

Since claim payments made in the same accounting year typically differ by large amounts over different DY, the observation of past claims inflation can be strongly biased if changes of the speed of finalization have been introduced by the companies. In fact a change of the CPCP observed in a given calendar year could be only partially a consequence of calendar time variations of the general claims cost level if an anticipation or a delay in the finalization of more costly claims took place. If some information is available on the historical speed of finalization, under appropriate assumptions some adjustment for these effects can be made on the observed claims inflation.

In this analysis we used a simple model for taking into account historical changes in speed of finalization. The model uses aggregate data by accounting year expressing the number of claims finalized in DY j as a fraction of the number of claims still open at the beginning of DY j. Under some simplifying assumptions a series of correction factors is provided suitable for adjusting the observed inflation factors of claims paid in the first DY. A detailed description of the model is provided in section B of the Appendix.

5.1.3 Choosing the time series of past inflation

In principle, the possibility exists that the choice of the weighting method in averaging past inflation rates over the companies introduces an undesirable distortion in the estimation of the required reserve, as a consequence of systematically different changes in projected losses for companies with different portfolio size.

To control this possible effect the triangles of paid losses of companies in the sample have been escalated using the different time series $\{r_i^{(x)}\}$ of inflation rates reported in table 5.1 and the corresponding best estimate of the overall losses $\overline{L}^{(x)}$ has been computed applying the chain-ladder algorithm to the inflation adjusted triangles. The percentage differences $\overline{L}^{(x)}/\overline{L} - 1$ with respect to the predicted values obtained by the unadjusted triangles have then been regressed over the statutory reserve R^s. In all the linear regressions the slope coefficient resulted not significantly different from zero. Over the different averaging criteria the r-squared ranged between 0.013 and 0.015. Given these results no systematic distortion effect on the expected OLL seems to be determined by the weighting method.

An identical analysis has been performed using as regressors the percentage differences $\mathbf{Pstd}(L^{(x)})/\mathbf{Pstd}(L) - 1$ and $\mathbf{Q}^{(75)}(L^{(x)})/\mathbf{Q}^{(75)}(L) - 1$, where the prediction errors were derived by the DFCL model and the quantiles were obtained assuming a lognormal underlying distribution. Also in these cases no significant correlation effect with the portfolio size has been detected.

5.1.4 Applying the stochastic models to inflation adjusted data

In the following applications the time series $\{r_i^{(P)}\}$ of the average inflation rates weighted with the total paid losses has been chosen as a proxy of the claims inflation. The corresponding average inflation rate over the observation period 1996-2004 is $r^{(P)} = 9.13\%$. The ten-year triangles of paid losses of the companies in the selected sample have been escalated using the inflation rates $\{r_i^{(P)}\}$ and both the ODP and the DFCL model have been applied to that data[2].

The chain-ladder algorithm has been applied to the inflation adjusted triangles, providing a new estimate $\overline{L}^*$ of the total undiscounted OLL for each company in the selected sample. The total BE values (in million Euros) for the 4 dimensional classes and for the overall sample are reported in table 5.2 and compared with the corresponding values from unadjusted triangles.

Class	R^s	$\overline{L}$	$\overline{L}^*$	$\overline{L} - \overline{L}^*$	(%)
1	15,114.43	14,116.69	11,216.05	2,900.64	20.55
2	7,257.39	7,604.49	6,033.41	1,571.09	20.66
3	1,541.70	1,556.26	1,233.59	322.68	20.73
4	128.45	135.65	108.97	26.68	19.66
total	24,041.97	23,413.09	18,592.01	4,821.08	20.59

Table 5.2: Projected OLL from inflation-adjusted paid losses and historical paid losses

On the overall sample it results a 20.59% reduction of OLL estimate due to the inflation adjustment of the historical data; the effect is roughly similar in the four classes.

5.2 Applying stochastic loss reserving models to inflation adjusted data

5.2.1 Projecting future paid losses under stochastic inflation

Future paid losses are essentially real in nature, therefore they are necessarily exposed to inflation. Since the past data have been escalated by past inflation, projected future payments can be considered as *nominal amounts*, that is do not include any rescaling for future inflation. Therefore a model for future inflation must be used to provide a reliable assessment of the OLL.

5.2.2 The stochastic inflation model

Future claims inflation has been modelled using a simple lognormal model. Let us denote by:

$$C'_{i,d_i+\tau}\,, \quad \tau = 1, 2, \ldots, n-1\,, \quad i = \tau+1, \ldots, n\,,$$

the r.v. representing the nominal value of future claim payment at the year-end τ for claims of AY i. The predictive distribution of $C'_{i,d_i+\tau}$ can be thought of as being provided by a specified model for loss reserving applied to inflation adjusted data. It is assumed that the effective claim payment is given by:

$$C''_{i,d_i+\tau} = C'_{i,d_i+\tau}\, p_\tau\,,$$

where the inflation index process p_θ ($\theta \geq 0$) is a geometric Brownian motion described by the stochastic differential equation:

$$dp_\theta = \mu\, p_\theta\, d\theta + \omega\, p_\theta\, dZ_\theta\,,$$

[2]From a regression analysis it resulted that an escalation of all payments at an annual inflation rate 9.13% flat can introduce dependencies of changes in the variability of projected payments on the company size. Hence it is important that the paid losses of each calendar year are escalated by the corresponding inflation factor.

with $p_0 = 1$. The process Z_θ is a standard Brownian motion independent of the claim development process, μ is the (continuously compounded) expected inflation rate and $\omega > 0$ is the volatility of the inflation index. The first two moments of p_τ at each year-end have the well-known expressions:

$$\mathbf{E}_0(p_\tau) = e^{\mu\,\tau}\,, \qquad \mathbf{Var}_0(p_\tau) = e^{2\,\mu\,\tau}\Big(e^{\omega^2\,\tau} - 1\Big)\,. \tag{5.1}$$

Remark. If the present value of the real payoff $C''_{i,d_i+\tau}$ is required the appropriate discounting rate must be used. In general the time zero price $\mathcal{V}(0; Y''_\tau)$ of a real payoff Y''_τ maturing at time $t = \tau$ must be derived under an arbitrage-free model for real interest rates. Examples can be found in [32]. ■

The inflation index can be applied to the total payments of each future year:

$$Y'_\tau := \sum_{i=\tau+1}^{n} C'_{i,d_i+\tau}\,, \quad \tau = 1, 2, \ldots, n-1\,,$$

in order to obtain the escalated values:

$$Y''_\tau = Y'_\tau\, p_\tau\,,$$

and the overall (undiscounted) OLL after inflation are given by:

$$L'' := \sum_{\tau=1}^{n-1} Y''_\tau\,.$$

5.2.3 Including the inflation process

By the independence assumption the stochastic inflation model can be easily incorporated in the typical loss reserving models.

Since the ODP model is applied by simulation the projected inflation can be simply included multiplying the simulated value $C'_{i,d_i+\tau}$ in each cell of the diagonal τ by a r.v. drawn by a lognormal distribution with mean and variance given by (5.1).

As concerning the DFCL model, it is easily shown that the appropriate expressions for the first two moments of the estimates of Y''_τ are:

$$\widehat{Y}''_\tau = \widehat{Y}'_\tau\, e^{\mu\,\tau}\,,$$

and:

$$\mathbf{Pvar}\big(\widehat{Y}''_\tau\big) = \Big[\mathbf{Pvar}\big(\widehat{Y}'_\tau\big) + \big(\widehat{Y}'_\tau\big)^2\Big]\, e^{2\,(\mu+\omega^2)\,\tau} - \big(\widehat{Y}'_\tau\big)^2\, e^{2\,\mu\,\tau}\,.$$

A closed form expression for the total prediction variance $\mathbf{Pvar}\big(\widehat{L}''\big)$ is rather cumbersome. A simple approximation can be obtained denoting by:

$$\mathbf{Pstd}^{(0)}\big(\widehat{L}'\big) := \sqrt{\sum_{\tau=1}^{n-1} \mathbf{Pvar}\big(\widehat{Y}'_\tau\big)}\,, \tag{5.2}$$

the uncorrelated total prediction error, and by:

$$\mathbf{Pstd}^{(1)}\big(\widehat{L}'\big) := \sum_{\tau=1}^{n-1} \mathbf{Pstd}\big(\widehat{Y}'_\tau\big)\,, \tag{5.3}$$

the perfectly correlated total prediction error. Since the expression $\mathbf{Pstd}\big(\widehat{L}'\big)$ of the effective (i.e. partially correlated) total prediction error is available, one can assume the approximate proportionality property:

$$\mathbf{Pstd}\big(\widehat{L}''\big) \approx \mathbf{Pstd}^{(0)}\big(\widehat{L}''\big) + c\left[\mathbf{Pstd}^{(1)}\big(\widehat{L}''\big) - \mathbf{Pstd}^{(0)}\big(\widehat{L}''\big)\right],$$

where:

$$c := \frac{\mathbf{Pstd}\big(\widehat{L}'\big) - \mathbf{Pstd}^{(0)}\big(\widehat{L}'\big)}{\mathbf{Pstd}^{(1)}\big(\widehat{L}'\big) - \mathbf{Pstd}^{(0)}\big(\widehat{L}'\big)}\,, \tag{5.4}$$

and where the expressions of $\mathbf{Pstd}^{(0)}\big(\widehat{L}''\big)$ and $\mathbf{Pstd}^{(1)}\big(\widehat{L}''\big)$ are the analogous of (5.2) and (5.3), respectively.

5.2.4 Results from models with stochastic inflation

The ODP and the DFCL model with stochastic inflation have been applied to the inflation adjusted triangles of paid losses of the companies in the selected sample assuming an annual rate of inflation of 7.5%, corresponding to a continuously compounded rate $\mu = \log 1.075 = 0.723$, and an inflation volatility $\omega = 0.04$.

Table 5.3 contains some preliminary results on the undiscounted OLL L'' obtained by the ODP model with inflation (also in this case a simplified notation has been used omitting the symbol "$\widehat{\ }$"). For each company in the selected sample we reported:

· the percentage difference between the best estimate $\overline{L}^*$ from inflation adjusted data without projected inflation and the best estimate $\overline{L}$ from historical data:

$$\Delta\overline{L}^* := \frac{\overline{L}^* - \overline{L}}{\overline{L}}\,;$$

· the percentage difference between the best estimate $\overline{L}''$ from inflation adjusted data with projected inflation and the best estimate $\overline{L}$ from historical data:

$$\Delta\overline{L}'' := \frac{\overline{L}'' - \overline{L}}{\overline{L}}\,;$$

· the percentage difference between the standard deviation of L'' and the standard deviation of L:

$$\Delta\mathbf{Pstd} := \frac{\mathbf{Pstd}(L'') - \mathbf{Pstd}(L)}{\mathbf{Pstd}(L)}\,;$$

· the percentage difference between the α-quantile of L'' and the α-quantile of L:

$$\Delta\mathbf{Q}^{(\alpha)} := \frac{\mathbf{Q}^{(\alpha)}(L'') - \mathbf{Q}^{(\alpha)}(L)}{\mathbf{Q}^{(\alpha)}(L)}\,.$$

As usual, companies are sorted by CodCv code. Also the average values are reported, the averages being weighted by the value of $\overline{L}$.

The same results obtained by the DFCL model with inflation are reported in table 5.4; also in this case companies are sorted by the coefficient of variation of the ODP model.

It results that the 20.59% decrease of the best estimate obtained by inflation adjusted data without projected inflation reduces to 3.83% assuming a future inflation at 7.5% p.a. flat. The assumption of inflation volatility at 4% p.a. seems rather conservative if one considers the effect on the OLL variability: on the average, the standard deviation of the OLL is increased by 39.35% in the ODP model and by 39.59% in the DFCL model. Consequently the required reserve decrease is reduced under the quantile approach. In the ODP model the decrease becomes 2.26% and 0.68% defining R^* as the 75-th and the 90-th percentile, respectively; the corresponding reductions for the DFCL are 2.60% and 1.35%.

The overall values by dimensional classes of the required reserve defined as mean, 75-th quantile and 90-th quantile of L'' are reported in tables 5.5 and 5.6 for the two models. The differences with respect to the corresponding values of R^* defined on the distribution of L derived from historical data are also given. As usual, values are expressed in million Euros.

CodCv	Class	$\Delta\overline{L}^{*}$ (%)	$\Delta\overline{L}''$ (%)	$\mathbf{\Delta Pstd}$ (%)	$\Delta\mathbf{Q}^{(75)}$ (%)	$\Delta\mathbf{Q}^{(90)}$ (%)
1	1	-19.79	-2.89	83.83	-1.11	0.95
2	1	-18.64	-2.89	57.98	-1.14	0.56
3	1	-20.01	-3.23	58.49	-1.50	0.38
4	1	-21.83	-4.83	45.19	-3.35	-1.99
5	2	-17.82	-3.49	36.83	-2.34	-1.13
6	2	-21.45	-3.70	51.49	-1.92	-0.30
7	1	-20.58	-3.22	52.11	-1.51	0.20
8	1	-16.88	-2.16	44.99	-0.40	1.46
9	2	-20.51	-3.28	41.01	-1.64	0.24
10	1	-19.82	-2.74	47.34	-0.79	1.30
11	1	-21.61	-4.99	23.39	-3.64	-2.76
12	2	-17.53	-2.34	33.81	-0.98	0.74
13	2	-23.18	-6.43	26.82	-5.04	-3.59
14	2	-22.10	-4.62	25.04	-3.47	-1.90
15	2	-16.67	-1.17	31.16	0.18	1.82
16	2	-22.71	-6.40	21.27	-5.17	-3.76
17	2	-23.41	-4.83	22.30	-3.55	-2.36
18	3	-20.03	-3.40	27.47	-2.05	-0.23
19	2	-21.26	-2.55	29.04	-0.95	0.71
20	3	-22.25	-4.82	24.19	-3.28	-1.59
21	2	-21.84	-3.40	25.64	-1.83	-0.10
22	2	-21.67	-5.67	17.13	-4.40	-3.05
23	2	-18.70	-3.32	21.65	-2.01	-0.54
24	2	-20.58	-4.78	14.93	-3.66	-2.48
25	3	-22.69	-4.83	16.04	-3.58	-2.35
26	3	-18.69	-2.85	15.17	-1.82	-0.58
27	3	-21.13	-5.78	9.63	-4.76	-3.81
28	3	-25.00	-5.75	12.84	-4.21	-3.25
29	3	-18.74	-2.09	23.92	-0.70	1.72
30	3	-17.87	-2.23	23.14	-0.54	1.36
31	3	-20.81	-3.12	23.96	-0.98	0.82
32	3	-20.75	-3.62	22.35	-1.68	0.41
33	4	-18.79	-0.17	20.24	1.35	3.35
34	1	-22.92	-4.89	8.42	-3.70	-2.82
35	2	-18.19	-1.73	35.15	1.61	4.58
36	3	-20.59	-4.61	25.49	-2.33	0.34
37	3	-19.81	-2.89	12.13	-1.77	-0.15
38	4	-19.13	-2.56	17.02	-0.60	1.53
39	4	-21.43	-5.07	8.54	-3.11	-1.66
40	3	-17.90	0.34	14.28	2.28	4.24
average		-20.59	-3.83	39.35	-2.26	-0,68

Table 5.3: ODP model with stochastic inflation - Differences w.r.t values without modelled inflation

CodCv	Class	$\Delta\overline{L}^*$ (%)	$\Delta\overline{L}''$ (%)	$\mathbf{\Delta Pstd}$ (%)	$\mathbf{\Delta Q}^{(75)}$ (%)	$\mathbf{\Delta Q}^{(90)}$ (%)
1	1	-19.79	-2.89	75.59	-1.59	-0.33
2	1	-18.64	-2.89	43.23	-1.74	-0.60
3	1	-20.01	-3.23	57.57	-1.62	-0.01
4	1	-21.83	-4.83	51.95	-3.52	-2.22
5	2	-17.82	-3.49	37.72	-2.60	-1.74
6	2	-21.45	-3.70	54.44	-2.42	-1.18
7	1	-20.58	-3.22	58.44	-1.68	-0.15
8	1	-16.88	-2.16	38.84	-0.97	0.22
9	2	-20.51	-3.28	34.61	-2.12	-0.96
10	1	-19.82	-2.74	48.39	-1.24	0.26
11	1	-21.61	-4.99	35.94	-3.88	-2.78
12	2	-17.53	-2.34	28.53	-1.28	-0.21
13	2	-23.18	-6.43	32.47	-5.08	-3.71
14	2	-22.10	-4.62	28.78	-3.63	-2.64
15	2	-16.67	-1.17	30.33	-0.05	1.07
16	2	-22.71	-6.40	21.12	-5.40	-4.38
17	2	-23.41	-4.83	8.77	-4.16	-3.45
18	3	-20.03	-3.40	25.46	-2.45	-1.49
19	2	-21.26	-2.55	32.55	-1.35	-0.15
20	3	-22.25	-4.82	27.66	-3.71	-2.60
21	2	-21.84	-3.40	28.03	-2.28	-1.15
22	2	-21.67	-5.67	17.86	-4.79	-3.89
23	2	-18.70	-3.32	17.55	-2.41	-1.48
24	2	-20.58	-4.78	9.09	-4.12	-3.42
25	3	-22.69	-4.83	17.20	-3.98	-3.12
26	3	-18.69	-2.85	14.58	-1.90	-0.88
27	3	-21.13	-5.78	4.23	-5.25	-4.69
28	3	-25.00	-5.75	9.04	-4.94	-4.07
29	3	-18.74	-2.09	27.18	-0.92	0.28
30	3	-17.87	-2.23	19.59	-1.02	0.29
31	3	-20.81	-3.12	20.41	-2.08	-1.00
32	3	-20.75	-3.62	31.19	-2.06	-0.42
33	4	-18.79	-0.17	21.01	0.79	1.79
34	1	-22.92	-4.89	8.20	-4.24	-3.56
35	2	-18.19	-1.73	36.16	0.94	4.23
36	3	-20.59	-4.61	22.32	-2.46	0.30
37	3	-19.81	-2.89	3.82	-2.28	-1.49
38	4	-19.13	-2.56	19.49	-1.11	0.56
39	4	-21.43	-5.07	21.53	-3.59	-1.96
40	3	-17.90	0.34	18.50	1.41	2.59
average		-20.59	-3.83	39.59	-2.60	-1.35

Table 5.4: DFCL model with stochastic inflation - Differences w.r.t values without modelled inflation

Class	R^s	$\overline{L}''$	$\mathbf{Q}^{(75)}(L'')$	$\mathbf{Q}^{(90)}(L'')$	$\Delta\overline{L}''$	$\Delta\mathbf{Q}^{(75)}(L'')$	$\Delta\mathbf{Q}^{(90)}(L'')$
1	15,114.43	13,585.94	14,345.86	15,136.87	-530.74	-317.12	-95.66
2	7,257.39	7,298.27	7,802.58	8,323.75	-306.25	-200.01	-73.29
3	1,541.70	1,499.35	1,643.87	1,800.98	-56.92	-35.15	-6.47
4	128.45	132.42	149.56	169.12	-3.22	-0.90	2.09
total	24,041.97	22,515.98	23,941.88	25,430.73	-897.11	-553.17	-173.33

Table 5.5: ODP model with stochastic inflation

Class	R^s	$\overline{L}''$	$\mathbf{Q}^{(75)}(L'')$	$\mathbf{Q}^{(90)}(L'')$	$\Delta\overline{L}''$	$\Delta\mathbf{Q}^{(75)}(L'')$	$\Delta\mathbf{Q}^{(90)}(L'')$
1	15,114.43	13,585.94	14,170.93	14,749.40	-530.74	-360.16	-180.99
2	7,257.39	7,298.27	7,685.44	8,083.37	-306.23	-225.16	-131.59
3	1,541.70	1,499.35	1,598.90	1,704.52	-56.92	-42.32	-24.52
4	128.45	132.42	142.66	153.64	-3.22	-1.60	0.45
total	24,041.97	22,515.98	23,597.93	24,690.93	-897.11	-629.24	-336.66

Table 5.6: DFCL model with stochastic inflation

Appendix A

Notations on moments and quantiles

Referring to a random variable (r.v.) X having the suitable regularity properties, the relevant summary statistics will be denoted as follows.

We shall denote by $\mathbf{E}_t(X)$ the expectation of X at time t (which is supposed to exist and to be finite). When the valuation date t is unambiguously identified we shall drop the suffix t and we also use the simplified notation $\overline{X} := \mathbf{E}(X)$. The variance of X is given by:

$$\mathbf{Var}(X) := \mathbf{E}[(X - \overline{X})^2]\,;$$

the corresponding standard deviations is $\mathbf{Std}(X) := \sqrt{\mathbf{Var}(X)}$.

For fixed $\alpha \in (0, 100)$ the α-th quantile $\mathbf{Q}^{(\alpha)}(X)$ of X is defined as the "lower quantile":

$$\mathbf{Q}^{(\alpha)}(X) := \inf\{x : \mathbf{P}(X \leq x) \geq \alpha\}\ .$$

If the distribution function $F(x) := (X \leq x)$ is continuous, one has $\mathbf{Q}^{(\alpha)}(X) = F^{(-1)}(\alpha)$.

Referring to the α-th quantile $Q^\alpha := \mathbf{Q}^{(\alpha)}(X)$, the corresponding expected shortfall of X is the expected value beyond quantile, that is:

$$\mathbf{S}^{(\alpha)}(X) := \mathbf{E}\big(X | X \geq Q^\alpha\big)\,.$$

Appendix B

A simple adjustment model for changes in speed of finalization

Let $n_{i,j}$ be the number of claims finalized in the DY j of AY i and denote by $A_{i,j}$ the number of claims of AY i which are still open at the beginning of DY j. One can define the *speed of finalization* in DY j for AY i as the ratio:

$$F_{i,j} := \frac{n_{i,j}}{A_{i,j}} ;$$

for fixed j, the speed of finalization $F_{i,j}$ is the fraction of outstanding claims of DY j paid in the accounting year $i + j - 1$. Denoting by N_i the ultimate number of claims of AY i, one has:

$$F_{i,j} := \frac{n_{i,j}}{N_i - \sum_{k=0}^{j-1} n_{i,k}} ,$$

with $n_{i,0} := 0$. We shall assume that for each AY i there are J different types of claims, with cost $c^{(k)}$ $(k = 1, 2, \ldots, J)$. These costs are supposed to be constant across CY (i.e. are not exposed to inflation). The fraction of claims of type k is denoted by:

$$\varphi_i^{(k)} := \frac{N_i^{(k)}}{N_i} ,$$

where $N_i^{(k)}$ is the ultimate number of type k claims. Of course $\sum_{k=1}^{J} N_i^{(k)} = N_i$ hence $\sum_{k=1}^{J} \varphi_i^{(k)} = 1$. The idea is that claims of type k are typically paid in DY k; moreover claims of type $k+1$ are usually more costly than claims of type k. As an example, the average costs $\overline{C}_j^*$ reported in figure 5.5 could be used as a proxy for type k costs (with $J = 13$). With these hypotheses an increase (decrease) in the speed of finalization $F_{i,j}$ corresponds to an increase of the number of claims of type $k > j$ $(k < j)$ paid in CY $i + j - 1$, thus causing an apparent inflationary (deflationary) effect in this year.

We are only interested in the cost of claims paid in the first DY. So we only consider:

$$F_{i,1} := \frac{n_{i,1}}{N_i} , \tag{B.1}$$

which represents the speed of finalization (of claims of the first DY) in the accounting year i. As we are referring only to the first DY, changes of $F_{i,1}$ can only produce an increased number of claims of type $2, 3, \dots, J$ to be paid in CY i. We assume that any acceleration of finalization can produce early payment of only type 2 claims. Hence in each CY claims in the first DY can only have cost at $c^{(1)}$ and $c^{(2)}$ level. Moreover we shall assume that the fraction of type 1 claims is constant across different AY; that is:

$$\varphi_i^{(1)} := \frac{N_i^{(1)}}{N_i} \equiv \varphi' , \quad i = 1, 2, \dots, n .$$

Let us denote by $n'_{i,1}$ and $n''_{i,1}$ the number of claims of type 1 and type 2, respectively, finalized in CY i, and DY 1. Of course:

$$n_{i,1} = n'_{i,1} + n''_{i,1} ; \tag{B.2}$$

moreover:

$$n'_{i,1} = \min\left\{N_i^{(1)}, n_{i,1}\right\} , \tag{B.3}$$

since the payment of claims of type 2 is made only after the payment of claims of type 1 is completed. Expression (B.3) can also be written as:

$$n'_{i,1} = \min\left\{\varphi' N_i, F_{i,1} N_i\right\} = N_i \min\left\{\varphi', F_{i,1}\right\} . \tag{B.4}$$

The total amount paid in AY i for claims of the first DY is:

$$c_{i,1} := n'_{i,1}\, c^{(1)} + n''_{i,1}\, c^{(2)} = c^{(1)} \left(n'_{i,1} + \rho\, n''_{i,1}\right) , \tag{B.5}$$

where $\rho := c^{(2)}/c^{(1)}$ is the ratio between the cost of type 2 and type 1 claims. If the costs of figure 5.5 are used one has $\rho \approx 2$. By (B.2) and (B.4) equation (B.5) can be written:

$$c_{i,1} = c^{(1)} \left[n'_{i,1} + \rho\left(n_{i,1} - n'_{i,1}\right)\right] = c^{(1)}\, N_i \left[\min\left\{\varphi', F_{i,1}\right\} + \rho\left(n_{i,1} - \min\left\{\varphi', F_{i,1}\right\}\right)\right] ;$$

hence the average cost per claim is given by:

$$\bar{c}_{i,1} := \frac{c_{i,1}}{n_{i,1}} = \frac{c_{i,1}}{F_{i,1}\, N_i} = c^{(1)}\, \frac{\min\left\{\varphi', F_{i,1}\right\} + \rho\left(F_{i,1} - \min\left\{\varphi', F_{i,1}\right\}\right)}{F_{i,1}} .$$

This expression can be used for deriving the cost escalating factor due to the annual change in speed of finalization, defined as:

$$q_{i,1} := \frac{\bar{c}_{i,1}}{\bar{c}_{i-1,1}} , \quad i = 2, 3, \dots, n .$$

We have:

$$q_{i,1} = \frac{\min\left\{\varphi', F_{i,1}\right\} + \rho\left(F_{i,1} - \min\left\{\varphi', F_{i,1}\right\}\right)}{\min\left\{\varphi', F_{i-1}\right\} + \rho\left(F_{i-1,1} - \min\left\{\varphi', F_{i-1,1}\right\}\right)}\, \frac{F_{i-1,1}}{F_{i,1}} . \tag{B.6}$$

The ratios $q_{i,1}$ can be used for correcting the observed inflation factors $f_{i,1} := \overline{C}_{i,1}/\overline{C}_{i-1,1}$ of claims paid in the first DY. Computing the inflation rates as $f_{i,1}/q_{i,1} - 1$ one should obtain a kind of "filtering" for the effect of changes in the speed of finalization.

The ratios (B.6) strongly depend on the level of φ', i.e. on the fraction of type 1 claims assumed to be generated in each AY. For a given time series:

$$\left\{F_{i,1},\ i = 1, 2, \dots, n\right\}$$

of speeds of finalization observed in n consecutive CY, the maximum values of the q factor (hence the maximum effects of changes in F) will be obtained for levels of φ' close to minimum observed value:

$$F^* := \min_i \left\{ F_{i,1} \right\}.$$

For $\varphi' \geq \max_i \left\{ F_{i,1} \right\}$ all the correction factors will be equal to 1. For $\varphi' << F^*$ a relevant fraction of type 2 claims is assumed as being normally paid in each year; so any change in the speed of finalization will have a reduced effect, producing a fairly constant series of $\bar{c}_{i,1}$ and consequently correction factors uniformly close to unit.

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