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ORIGINAL RESEARCH PAPER

Optimal management of immunized portfolios

Riccardo Cesari[1,2,3] · Vieri Mosco[2,3]

Abstract We generalize the contribution of Fong and Vasicek (Financ Anal J 39:73–78, 1983a; Innov Bond Portf Manag Durat Analy Immun 1983:227–238, 1983b; J Financ 39:1541–1546, 1984) by developing a risk-return optimization problem for immunized life insurance portfolios. The M^2 measure of risk for arbitrary changes of the term structure of interest rates is used for a bond portfolio with duration matched to a given liability horizon. The immunization case becomes a "passive" strategy among an entire menu of active management decisions in which a partial risk minimization is exchanged for more return potential. As in the classical Markowitz (Portfolio Selection, New York, Wiley, 1959) approach, an efficient frontier at the given horizon provides the optimal tradeoff between risk and return. An empirical application to insurance companies shows that such a perspective may be proved useful to highlight which segregated funds can be re-positioned over the efficient frontier, at the chosen level of the firm's risk appetite.

Keywords Insurance companies · Immunization · Convexity · Fong–Vasicek theorem · Efficient frontier

JEL Classification G22 · G11 · G12

✉ Riccardo Cesari
  riccardo.cesari@unibo.it

1   University of Bologna, Bologna, Italy

2   IVASS, Rome, Italy

3   IVASS, Research Department, Rome, Italy

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1 Introduction

Notwithstanding the strong innovation wave which is hitting the insurance industry, a large share of its business model still relies on the good old principle of selling guarantees through a maturity matching between assets and liabilities. This matching and the general approach known as integrated management of assets and liabilities (ALM) has essentially the aim to cope with the interest rate risk, i.e. the risk of an asymmetric impact of interest rate movements (and risk factors in general) to the asset and the liability side of the balance sheet. As an alternative to the basic cash flow matching solution, the duration concept, as suggested by Redington [24], provided a feasible, more sophisticated, but more vulnerable solution. In his seminal work, he introduced the term of immunization to indicate a balance sheet not exposed to interest rate movements. "Hedging" would be the modern equivalent. The essential aspect is the link between duration D (or average of cash flow times) and the price sensitivity to interest rates:

$$D = -\frac{1}{P}\frac{\partial P}{\partial r}$$

so that the change in portfolio value due to a change in interest rates is proportional to portfolio duration. In a balance sheet set up, the risk capital is $E = A - L$ and we obtain, using the duration formula:

$$\frac{\partial E}{\partial r} = \left(-\frac{1}{L}\frac{\partial L}{\partial r}\right)L - \left(-\frac{1}{A}\frac{\partial A}{\partial r}\right)A = \left(D_L - D_A\frac{A}{L}\right)L$$

where the interest rate sensitivity of risk capital is a function of four components: durations of asset and liability, leverage A/L and liability level L.

If $E > 0$ we have the formula for the equity duration:

$$D_E = D_L + \frac{A}{E}\cdot\left(D_A - D_L\right) \gtreqless 0$$

where D_E can be regarded as the duration of liability plus the product of leverage A/E and mismatch.[1] If $E \equiv 0$ ($A = L$. or zero equity portfolio), we obtain $D_A = D_L$ and this perfect matching represents the fundamental immunization formula. However, the duration is a reliable measure of risk only in case of special interest rate changes (small, additive shift of the term structure): as shown by Fong and Vasicek [11], arbitrary movements of rates have an impact which depends not only on the average of cash flow times (D), but also on their dispersion (M^2.). In this broader sense, immunization as a minimum (zero) risk approach must be considered only

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[1] See Messmore [20] who shows that if $E > 0$ (the present value of assets greater than the present value of liabilities) the immunization of E, i.e. $D_E = 0$, implies $D_A < D_L$. In the following we assume a fixed income portfolio with $A = L$.

as a special, one-dimensional case of the portfolio optimization problem, where, as in the classical Markowitz model [18], an entire risk-return frontier provides a complete menu of management decisions, according to the firm's propensity to risk (risk appetite) and financial market opportunities. Fong and Vasicek [9, 10] introduced this double, risk-return dimension into the immunization literature. Generalizing their approach, we shall provide an implementable model according to which an insurance company could estimate the efficient frontier facing its fixed income balance sheet and evaluate, against arbitrary interest rate movements, how distant its portfolio is from efficiency and how much risk it is bearing with the chosen asset–liability mix.

In the next paragraph we shall review a few basic results of the classical theory of immunization, particularly useful for our developments. Then we will see in practice the term structure movements as suggested by recent European stress tests and financial market dynamics. In paragraphs 4 and 5 we will analyze passive and active strategies of portfolio immunization providing, in paragraph 6, an empirical example of the model. The final section concludes the paper with a recap of results and possible extensions.

2 Some results of the classical theory of immunization[2]

Let A_j for $j = 1, \ldots, m$ be the cash flows of a bond portfolio, available at time s_j, $j = 1, \ldots, m$, $0 < s_1 < \cdots < s_m$. Let $P(0, s)$ be the initial (time 0) discount function for maturity s.

The current value of the assets is:

$$A(0) = \sum_{j=1}^{m} A_j P(0, s_j)$$

where

$$P(0, s) = \exp\left(-\int_0^s r_{FW}(0, \tau) d\tau\right) = \exp\left(-sR(0, s)\right)$$

and $r_{FW}(0, \tau)$ is the initial instantaneous forward rate:

$$r_{FW}(0, \tau) \equiv -\frac{\partial \ln P(0, \tau)}{\partial \tau}$$

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[2] See Appendix 1 for a proof of the theorems. Without loss of generality we assume continuous compounding.

and $R(0, s)$ is the initial spot rate for maturity s:

$$R(0, s) = \frac{1}{s} \int_0^s r_{FW}(0, \tau) d\tau$$

Let $\bar{L}$ be the value of a single liability at the target time H, such that[3]:

$$A(0) = L(0) = \bar{L} \exp\left(-\int_0^H r_{FW}(0, \tau) d\tau\right). \text{(budget constraint)}$$

Note that $\bar{L} \equiv A_0(H)$ is the forward, time H value of the portfolio calculated at time 0:

$$\bar{L} \equiv A_0(H) = A(0)\exp\left(\int_0^H r_{FW}(0, \tau) d\tau\right) = \sum_{j=1}^m A_j \exp\left(\int_{s_j}^H r_{FW}(0, \tau) d\tau\right)$$

If forward interest rates do not change, the value of the bond portfolio at the target date is $A(H) = \bar{L}$. If, instead, interest rates do change we could have $A(H) \lesseqgtr \bar{L}$.

In fact:

$$A(H) = \sum_{j=1}^m A_j P(H, s_j) = \sum_{j=1}^m A_j \exp\left(\int_{s_j}^H r_{FW}(H, \tau) d\tau\right) = A_H(H)$$

The balance sheet is said to be immunized against interest rate movements if $A(H) \geq \bar{L}$ and it would be interesting to find the conditions (if any) under which a given balance sheet has this property. Clearly, the immunizing conditions depend on the assets and liabilities as well as the type of movements assumed for the interest rates.

Theorem 1 (constant shift) *If the term structure of interest rates undergoes an additive and small constant shift from $r_{FW}(0, \tau)$. to $r_{FW}(0^+, \tau) = r_{FW}(0, \tau) + \Delta_0$ then the balance sheet with liability $\bar{L}$ at time H (or the portfolio at the target horizon H) is immunized if the duration $D(A)$ of the portfolio is equal to H.*

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[3] This multiple asset—single liability case is equivalent to the case of an asset only bond portfolio with target horizon H and target value $\bar{L} = A(0) \exp\left(\int_0^H r_{FW}(0, \tau) d\tau\right)$.

This result is known as Fisher and Weil [8] theorem. In simple words, the duration condition guarantees that, after the assumed constant shift, the gain (loss) obtained through the reinvestment effect of short term cash flows is not smaller (not greater) than the loss (gain) obtained via the price effect of the long term asset component of the portfolio. The special case of a flat term structure applied to the balance sheet of the life business of an insurance company (with multiple liability cash flows) had already been obtained by Redington [24], who introduced also the term immunization "*to signify the investment of the assets in such a way that the existing business is immune to a general change in the rate of interest*" (ivi, p. 289). Samuelson [25] provided a similar proof with application to the US banking system. Redington's "mean term" and Samuelson's "average time period" can be acknowledged as the Macaulay [17] duration.[4] Note that the duration condition must be maintained over time, so that immunization is in fact a dynamic process.

In terms of rate of returns with continuous compounding, $e^{H\bar{R}_0(H)} \equiv \frac{\bar{L}}{L(0)} \equiv \frac{A_0(H)}{A(0)}$

so that $\bar{R}_0$ is the ex-ante target rate of return evaluated at time 0, also obtained ex-post in case of stable interest rates. Taking logs and differentiating:

$$H\bar{R}_0(H) \equiv \ln\left(A_0(H)\right) - \ln\left(A(0)\right)$$

$$\Delta\bar{R}_0 = \frac{1}{H}\frac{\Delta A_0(H)}{A_0(H)}$$

and, under the Fisher–Weil hypotheses, $\Delta\bar{R}_0 \geq 0$ for any small, additive shift of the term structure.

The Fisher–Weil theorem has been generalized in terms of a non-constant shift [11, 22, 27, 29].

Theorem 2 (non-constant shift)   *If the term structure of interest rates undergoes an additive and non-constant shift from $r_{FW}(0, \tau)$ to $r_{FW}\left(0^+, \tau\right) = r_{FW}(0, \tau) + \Delta_0(\tau)$ then the balance sheet with liability $\bar{L}$ at time H (or the portfolio at the target horizon H) is immunized if the duration D(A) of the portfolio is equal to H (duration matching condition) and $\Delta_0^2(\tau) \geq \Delta_0'(\tau)$ for every τ ("convexity condition").*

Note that $\Delta_0(\tau)$ is the term structure of the change in the forward rates $r_{FW}(0, \tau)$ so that $\Delta_0'(\tau)$ is the slope of this change and the convexity condition is satisfied if $\Delta_0(\tau)$ is monotone decreasing (the longer the maturity, the smaller the changes). More complex cases are given in Appendix 2. In general, if the term structure shift is not convex, the change in the forward value will be negative. The total effect is the sum of a "shift in level" component or convexity effect, always positive, Δ^2, and a "slope of shift" component Δ', or risk effect, of ambiguous sign. Under the traditional approach of a constant, parallel shift of the term structure ($\Delta = \text{const}, \Delta' = 0$)

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[4] The duration measure was independently obtained by Hicks [13] with the name of "average period". Earlier developments could be seen in Lidstone [16].

the total effect is due to convexity only and it is always positive, so that a high convexity portfolio would gain an extra-return.[5]

A Corollary of Theorem 2 gives a useful approximation.

Corollary of Theorem 2 *The following second-order approximation holds:*

$$\frac{\Delta A_0(H)}{A_0(H)} \equiv \frac{G(r_{FW} + \Delta_0)}{\bar{L}} \simeq \frac{1}{2} M_0^2(D)[\Delta_0^2(D) - \Delta_0'(D)]$$

where $M_0^2(D)$ is a variability measure of portfolio cash-flow times measured at time 0:

$$M_0^2(D) \equiv \frac{\sum_{j=1}^{m} a_j (s_j - D)^2}{\sum_{j=1}^{m} a_j} = \frac{\sum_{j=1}^{m} A_j (s_j - D)^2 \exp\left(-\int_0^{s_j} r_{FW}(0, \tau) d\tau\right)}{\sum_{j=1}^{m} A_j \exp\left(-\int_0^{s_j} r_{FW}(0, \tau) d\tau\right)}$$

A lower bound for the change in value was obtained by Fong and Vasicek [11].

Theorem 3 (lower bound for non-constant shift) *If the term structure of interest rates undergoes an additive and non-constant shift from $r_{FW}(0, \tau)$ to $r_{FW}(0^+, \tau) = r_{FW}(0, \tau) + \Delta_0(\tau)$ and if the duration $D(A)$ of the portfolio is equal to the maturity H of the liability $\bar{L}$ (duration matching condition), then a lower bound for the change in value of the portfolio is:*

$$\frac{\Delta A_0(H)}{A_0(H)} \equiv \frac{G(r_{FW} + \Delta_0)}{\bar{L}} \geq -\frac{1}{2} K_0 M_0^2(D)$$

where K_0 is the maximum of $\Delta_0'(\tau)$ and $M_0^2(D)$ is the time 0 variability of portfolio cash-flow times.

Note that M^2 is a proper variance of times, given that D is the time average (duration). This implies $M^2 = C - D^2$ where C is the asset convexity (average of times squared).[6]

The extension to multiple liabilities was firstly considered in Redington's seminal paper. He expressed financial immunization by introducing (sufficient) conditions for a fixed income portfolio backing insurance commitments (spread across many

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[5] Clearly, interest rate dynamics implying constant shifts are not arbitrage-free. See Boyle [3].

[6] As duration is linked to the first derivative of the price with respect to the interest rate, convexity represents the second derivative. It is easy also to show that $M^2 = -\frac{\partial D}{\partial r}$.
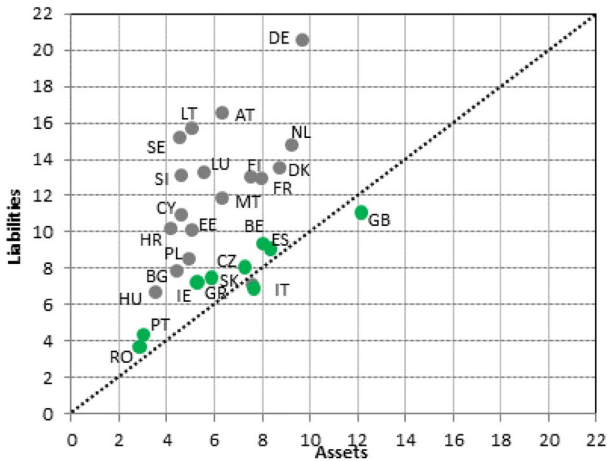
Fig. 1 Duration of assets and liabilities in the EIOPA stress test report

maturities) to be immunized (i.e. protected) against parallel and small variations of a flat term structure. He proved that assets remain above liabilities in value if, subject to the initial balance constraint, the duration (time average) of asset cash flows equals the duration of liabilities and the convexity (time dispersion) of asset cash flows is greater than that of liability cash flows.[7]

3 Some market data

Depending on their asset and liability composition, insurance companies will be differently affected by future levels and shapes of interest rates. The duration mismatch gives an approximate measure of this exposure: as shown in Fig. 1, country averages in Europe are widely dispersed, often far away from the perfect matching line.[8] This implies that market movements could impair the firms by different amounts, depending also on the different dynamics that will take place in real markets. However, as we shall see, even well matched countries, like Italy, Great Britain, Belgium, Spain and Portugal, are not immunized against general movements in the shape of the term structure.

For example, in its 2014 stress test exercises, EIOPA, the European Supervisory Authority for Insurance and Pension Funds, paid special attention to a scenario of lowering rates, with a long-lasting flattening view of the term structure (so called Japanese scenario).[9] In Fig. 2 this low yield scenario is depicted along with the

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[7] This multiple liability immunization has been generalized by Shiu [28].

[8] The EIOPA calculation of the liability side in Fig. 1 does not take into account all the contract optionality available in different countries so that the country relative positions could be altered.

[9] For instance, see the EIOPA official presentation of the 2014 Stress Test in EIOPA [6].
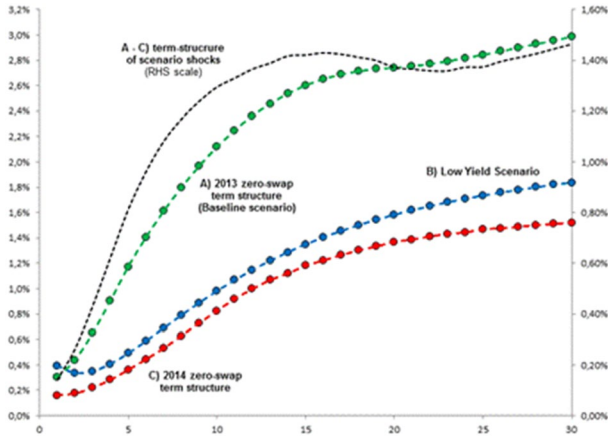
Fig. 2 EIOPA 2014 stress test scenarios for European interest rates

actual market term structure at the end of 2013 ("Baseline scenario") and that at the end of 2014, which resulted even lower than envisaged before.

One can see that the hypothetical and the actual market movements are far from being a small, constant shift of the interest rates: the term structure of the shocks $\{\Delta(\tau), \tau = 1, 2, \ldots, 30\}$ is both non infinitesimal and not flat across maturities, quite far from the typical 100 basis points figure, ranging roughly from + 20 bps to − 120 bps. According to the EIOPA 2014 Stress Test Report [7], the results of the stress exercise have been evaluated with respect to "*the size of duration mismatches between assets and liabilities as well as mismatches in internal rate of return of assets and liabilities*" which are "*considered the main drivers for the severity of an interest rate stress*".[10] However, as it is well known after Redington [24] and the following literature,[11] in case of not small and not parallel shocks to the term structure, the classical "duration gap" analysis should be generalized to take into account also portfolio convexity and its effect on the balance sheet.[12]

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[10] See EIOPA [7], section C, "Low Yield Module Description and Results", paragraph 2, sub-paragraph 56.

[11] See Martellini, Priaulet and Priaulet [19], ch. 6 and the references therein.

[12] Apart from EIOPA Stress Tests, convexity effects are still ignored in many empirical applications as well as popular textbooks in banking and finance (e.g. Mishkin and Apostolos [21]). Moreover, under the Basel III Capital Requirement Regulation (CRR, in force since 1 January 2014), the Standardised Approach includes a calculation for the own funds requirement for the general risk on debt instruments only based on the duration. See European Banking Authority (EBA), *Interactive Single Rulebook*, article 340 in https://www.eba.europa.eu/regulation-and-policy/single-rulebook. A partial justification can be found in the relevance of the first order effect (duration) as documented in Schaefer [26] and in the empirical literature on Principal Components (Martellini, Priaulet and Priaulet [19], ch. 3) where the first factor (the parallel shift of the term structure) accounts for 50% to 90% of total variability of interest rate changes, especially in the long side of the yield curve.

4 M^2 and the passive strategy of minimizing risk

Following Fong and Vasicek [9, 11], let $\bar{R}_0$ be the guaranteed target return, fixed ex ante at time 0, for the investment horizon (holding period) H. If the portfolio duration D is equal to H, the Fong–Vasicek lower bound (Theorem 3 above) shows that the ex post portfolio value has a minimum percentage change (maximum shortfall):

$$\frac{\Delta A_0(H)}{A_0(H)} \geq -\frac{1}{2} \underbrace{M_0^2}_{endogenous} \underbrace{\max_{\tau} \left\{ \Delta_0'(\tau) \right\}}_{exogenous}$$

where M_0^2 is a variance of times to payment and $max\{\Delta_0'\}$ is the maximum change of the slope of the current term structure across its maturities. By definition:

$$M_0^2 = \sum_{j=1}^{m} \left(s_j - D \right)^2 \cdot w_j$$

where D is the duration and the weights w_j are given by the present values of the cash flows at time s_j. As duration represents a time average, M_0^2 is a time variance and can be expressed in terms of duration and convexity. Note that, according to the Fong and Vasicek lower bound, M_0^2 is a risk measure in that it captures the exposure to any arbitrary movement of the discount curve. It suffices, therefore, to minimize M_0^2 in order to minimize such exposure, maximizing, at the same time, the lower bound for changes in the forward value of the portfolio. If the absolute minimum of $M_0^2 = 0$ is attainable (using zero-coupon bonds of maturity H), then $\frac{\Delta A_0(H)}{A_0(H)} \geq 0$ and the forward portfolio value cannot undergo capital losses. This prudential strategy of minimizing M_0^2 might be regarded as a "passive" management, in the sense that it collapses into a simple one-dimensional recipe: buy the minimum risk portfolio and stick to it.

An example could be useful: let us consider the case of two portfolios, the "bullet" and the "barbell". A perfect "bullet" portfol with maturity D has a minimum M_0^2 of zero. In practice, it is composed by low-coupon securities with maturities close to D so that M_0^2 is close to zero. Viceversa, a "barbell" portfolio is a set of very short and very long securities with large M_0^2 even if it has the same duration D. These two portfolios with equal duration D are differently affected by a positive twist of interest rates ("steepening"), i.e. a decrease in short rates and an increase in long rates. In fact, this twist has effects both in terms of "reinvested income" and "capital gain" because short term rates have become lower, producing lower coupon (higher cost prices) from reinvested income whilst long term rates are now higher, producing higher capital losses realized at the horizon. This will produce a shortfall of the portfolio value with respect to the target value (a negative twist will produce the opposite). But the two portfolios are differently (negatively) affected: the barbell portfolio (with higher M_0^2) is penalized

twice, because, due to the barbell profile, i) a greater number of securities/cash flows reach sooner the maturity date with reinvestment at lower interest rates (due to the downward shock on short interest rates) and ii) a higher and longer portion of the portfolio will be outstanding after the horizon date, producing a higher capital loss, given that securities with longer maturities will depreciate more heavily. This means that both M_0^2 and the risk of the barbell portfolio are greater than those of the bullet.

5 Active strategy: the risk-return tradeoff optimization

Immunization, as a minimum (zero) risk approach, must be considered as a special case in the broader area of portfolio optimization, in which risk-return considerations provide an entire menu of management decisions, depending on the firm's risk aversion and financial market opportunities. In the following, starting from Fong and Vasicek [9, 10], we provide this generalized risk-return approach for duration matching, insurance portfolios, analogous to Markowitz's mean–variance asset portfolio analysis.

Let $\bar{R}_0(H)$ be the current annual ex-ante rate of return over the horizon H. By definition:

$$e^{H\bar{R}_0(H)} \equiv \frac{A_0(H)}{A(0)}$$

In case of no shift of the term structure, the ex-ante is also the ex-post rate of return.

Using the Corollary of Theorem 2, after an instantaneous non-constant shift of the term structure we have the second-order approximation:

$$\Delta\bar{R}_0(H) = \frac{1}{H}\frac{\Delta A_0(H)}{A_0(H)} \simeq \frac{1}{H}M_0^2(H)\Delta S_0(H)$$

where:

$$\Delta S_0(H) \equiv \frac{1}{2}\left[\Delta_0^2(H) - \Delta_0'(H)\right] \gtreqless 0$$

is a special function of the term structure shift from time 0 to time 0^+. and maturity H (see Appendix 2). Note that this function is the sum of a "shift in level" component ("convexity effect"), always positive, Δ^2, and a "slope of shift" component Δ', ("risk effect") of ambiguous sign. In case of adverse shift ($\Delta_0^2 < \Delta_0'$, $\Delta S_0 < 0$)he realized return will be under the target value.

The portfolio risk can be measured by the volatility (standard deviation) of $\Delta\bar{R}_0(H)$ and it is proportional to the dispersion measure M_0^2. As explained by Fong and Vasicek [10, p. 235]: "*A portfolio with half the value of M^2 than another portfolio can be expected to produce half the dispersion of realized returns around the target value when submitted to a variety of interest rate scenarios than the other portfolio*".

Summing up all the shape changes between 0. and $H - 1$. we have:

$$R_H(H) - \bar{R}_0(H) = \sum_{t=0}^{H-1} \Delta R_t(H) \simeq \frac{1}{H} \sum_{t=0}^{H-1} M_t^2(H) \Delta S_t(H)$$

where the difference, $R_H - \bar{R}_0$, between the future annual rate $\frac{1}{H} ln\left(\frac{A_H(H)}{A(0)}\right)$ and current ex-ante rate is a random variable (sum of H random variables $\Delta S_t(H)$), with mean and variance given by:

$$\mu_H \equiv E(R_H - \bar{R}_0), \ \sigma_H^2 \equiv Var(R_H - \bar{R}_0).$$

In practice, in all financial markets, cash flows can be typically bought and sold only in pre-defined "bundles" (the coupon bonds) so that optimal management must be set in terms of available bond portfolios. Given a set of K different bonds, the portfolio return can be represented as a linear combination of K random variables:

$$\sum_{i=1}^{K} w_i(R_{iH} - \bar{R}_{i0})$$

with mean μ_{iH} and covariance

$$\sigma_{ij} = Cov(R_{iH} - \bar{R}_{i0}, R_{jH} - \bar{R}_{j0})$$

Clearly:

$$R_{iH} - \bar{R}_{i0} = \frac{1}{H} \sum_{t=0}^{H-1} M_{it}^2(H) \Delta S_t(H) \quad i = 1, \dots, K$$

and we link the dynamics of M_{it}^2 and ΔS_t to their current values M_{i0}^2 and ΔS_0 as follows (see [10]):

$$M_{it}^2(H) \Delta S_t(H) \cong M_{i0}^2(H)\left(\frac{H-t}{H}\right)^3 \Delta S_0(D_{i0})\left(\frac{H-t}{D_{i0}}\right)$$

where, if D_{0i} is the i-th bond duration, we have[13]:

$$M_{i0}^2(H) = \sum_{j=1}^{m}(s_j - H)^2 w_{0j} = M_0^2(D_{i0}) + (D_{i0} - H)^2$$

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[13] Note that using (realistically) bonds instead of cash flows has no effect on the calculation of duration but it affects the calculation of risk. In fact, the duration of a portfolio of bonds is simply the "portfolio" (average) of bond durations; however, the M^2 of a portfolio of bonds is the "portfolio" of M^2 (within variance) plus the variance of durations (between variance). See the Appendix 4.

so that[14]:

$$\mu_{iH} = E(R_{iH} - \bar{R}_{i0}) = \frac{1}{H} \sum_{t=0}^{H-1} M_{it}^2(H) E(\Delta S_t(H)) = \frac{1}{H} M_{i0}^2(H) E(\Delta S_0(D_{i0})) \frac{1}{D_{i0}} \sum_{t=0}^{H-1} \frac{(H-t)^4}{H^3},$$

$$\sigma_{ij} = Cov(R_{iH} - \bar{R}_{i0}, R_{jH} - \bar{R}_{j0}) = Cov\left(\frac{1}{H} \sum_{t=0}^{H-1} M_{it}^2(H) \Delta S_t(H), \frac{1}{H} \sum_{v=0}^{H-1} M_{jv}^2(H) \Delta S_v(H)\right)$$

$$= \frac{1}{H^2} M_{i0}^2(H) M_{j0}^2(H) \left(\sum_{t=0}^{H-1} \frac{(H-t)^4}{H^3}\right)^2 \frac{1}{D_{i0}} \frac{1}{D_{j0}} Cov(\Delta S_0(D_{i0}), \Delta S_0(D_{j0}))$$

The active strategy is to find the portfolio w_i maximizing the objective function U, in mean and variance:

$$\max_{w_i} U(w'\mu, w'\Sigma w)$$

$$\begin{cases} \sum_{1 \le i \le K} w_i \cdot D_{i0} = H \text{ (target duration constraint)} \\ \sum_{1 \le i \le K} w_i = 1 \text{ (budget constraint)} \\ w_i \ge 0 \text{ (no short selling)} \end{cases}$$

where μ is the vector of expected returns μ_{iH} and $\Sigma \equiv [\sigma_{ij}]$ is the covariance matrix.

As an example: $U(w'\mu, w'\Sigma w; \varphi) = w'\mu - \varphi \sqrt{w'\Sigma w}$

where the confidence level parameter φ (implying the maximization of the lower bound of the return confidence interval) has the same role of the risk aversion parameter in Markowitz risk-return approach: the higher the parameter, the more relevant the effect of portfolio variance and the more conservative the preferred portfolio will be. For $\varphi = 0$ the optimal portfolio will maximize the ex ante rate of return; for φ large, the optimal portfolio is the passive portfolio with minimum immunizing risk. As in the classical Markowitz approach, the solution procedure can be divided in steps, the first one being the minimum variance portfolio for a given ex ante differential return m and the given horizon H:

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[14] Note that Fong and Vasicek [10] assume, in the variance calculation, the asymptotic approximation:
$\sum_{t=0}^{H} M_t^4 = \frac{M_0^4}{H^6} \sum_{t=0}^{H}(H-t)^6 = \frac{M_0^4}{H^6} \frac{H(H+1)(2H+1)(3H^4+6H^3-3H+1)}{42} \xrightarrow[for H \to \infty]{} \frac{H}{7} M_0^4$

so that the variance is proportional to $\frac{M_0^4}{7H}$. We do not use this asymptotic approximation.

$$\min_{w_i} \sqrt{w'\Sigma w}$$

$$
\begin{cases}
\displaystyle\sum_{1 \leq i \leq K} w_i\, \mu_{iH} = m \\
\displaystyle\sum_{1 \leq i \leq K} w_i \cdot D_{i0} = H \text{ (target duration constraint)} \\
\displaystyle\sum_{1 \leq i \leq K} w_i = 1 \text{ (budget constraint)} \\
w_i \geq 0 \text{ (no short selling)}
\end{cases}
$$

By varying m, the efficient frontier $m(\sigma)$ can be traced, for a given H. More generally, assuming the horizon H. as an additional control variable, an efficient surface $m(\sigma, H)$ could be obtained, showing the portfolio possibilities for an "active" management strategy in terms of optimal return-risk-horizon tradeoff.

6 An empirical application

We provide an empirical application using a sample of Italian data. In Italy, at the end of 2014, 56 life insurance companies totalled 429 billion euros of segregated funds representing with-profit, participating policies,[15] mainly invested (218 billion euros) in fixed coupon bonds.

In a sample of 10 medium-large life insurance companies (covering a market share of 54% in terms of collected premiums) the duration gap is particularly small, as required. In 5 cases the duration was about 7.5 years and in the other 5 cases it was about 10 years. For simplicity, the Government bond market was reduced into four bond benchmarks with maturity 3, 5, 10 and 20 years respectively (see Appendix 3), so that the results cannot measure the "selection" effect of asset management. The following frontiers (Fig. 3) have been obtained by calculating, as explained in the previous paragraph, the efficient frontiers with horizon (average duration) $H = 7.5$ and $H = 10$ respectively. The gap between each firm and the frontier is a measure of the "allocation" inefficiency of the firm's portfolio.

A firm finding its portfolio appreciably distant from the efficiency curve could reduce the level of risk without penalty in the expected return dimension or, vice-versa, reallocate its assets to reach a higher level of expected return while keeping up its risk exposure. In the first case, it is plausible that it has to reduce the portfolio "barbellness" by increasing the share of bonds with a lower M^2 and/or the share of zero-coupon bonds of the given maturity H. In the second case, it is probable that a few coupon bonds are available in the market with better return perspectives and

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[15] Because of the profit participation mechanism, the liability cash-flows partly depend on the asset cash-flows, through the returns realized by the assets in the segregated funds. Usually, this effect is negligible.
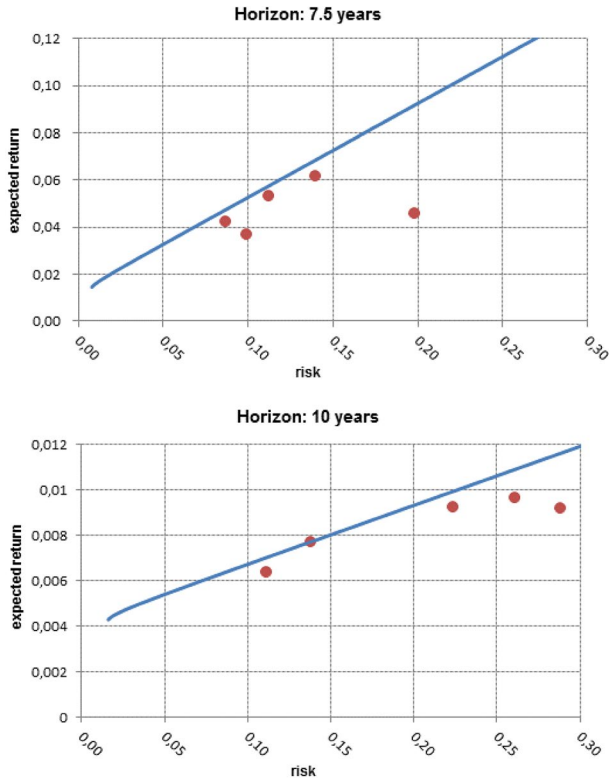
Fig. 3 Efficient frontiers for immunized portfolios at different horizons

better (de-) correlation for given term structure movements. The reallocation in favor of these bonds would improve the portfolio expected return without increasing its risk.

7 Conclusions and further developments

Following the seminal work by Fong and Vasicek [9–11], it is possible to actively manage the "immunization risk" of a duration-matching portfolio, i.e. a portfolio which, notwithstanding the equality of asset and liability durations (immunized portfolio), has a risky return, being exposed to the effects of arbitrary shifts of the term structure of interest rates. This risk is proportional to a measure (M^2) of the dispersion of the cash flow dates around the duration average so that, in this risk-return framework, you can build a generalized optimal management of the company's bond portfolio essentially in the same way as asset managers choose the optimal asset portfolio along Markowitz's efficient frontier. The empirical application of the model shows that it can disclose how distant the actual portfolio is

from the efficient frontier, helping to select, along the frontier, the optimal bond portfolio corresponding to the firm's risk appetite.

Extensions of the analysis are manifold. Theoretically, one can attempt to consider the multiple-liability framework in order to be more consistent in the modeling of the life insurance business. Moreover, the shocks to the forward rates can be assumed to have an explicit stochastic dynamics, to be exploited in the calculation of moments. At the same time, the empirical implementation could take into account the complete set of bonds available in the financial market, in order to have more refined measures of risks, returns and the frontier. The calculation of a three-dimensional frontier (risk, return and horizon) could be implemented through this more refined modelling and it could be proved useful in generalizing the traditional profit testing activity for new insurance products.

Appendix 1: Proof of the theorems

Proof of Theorem 1 The current budget constraint expressed at the forward time H is:

$$\sum_{j=1}^{m} A_j \exp\left(\int_{s_j}^{H} r_{FW}(0,\tau)d\tau\right) \equiv \sum_{j=1}^{m} a_j = \bar{L}$$

Let

$$G(r_{FW} + \Delta_0) \equiv \sum_{j=1}^{m} A_j \exp\left(\int_{s_j}^{H} (r_{FW}(0,\tau) + \Delta_0)d\tau\right) - \bar{L} \equiv \Delta A_0(H)$$

We have:

$$\frac{\partial G}{\partial \Delta_0} = \sum_{j=1}^{m} A_j(H - s_j) \exp\left(\int_{s_j}^{H} (r_{FW}(0,\tau) + \Delta_0)d\tau\right)$$

$$\frac{\partial^2 G}{\partial \Delta_0^2} = \sum_{j=1}^{m} A_j(H - s_j)^2 \exp\left(\int_{s_j}^{H} (r_{FW}(r,\tau) + \Delta_0)d\tau\right) > 0$$

so that $G(r_{FW} + \Delta_0) = 0$ for $\Delta_0 = 0$ and $G(r_{FW} + \Delta_0) \geq 0$ for every Δ_0 in a neighbourhood of 0, if $\Delta_0 = 0$ is a (local) minimum i.e. if:

$$\frac{\partial G}{\partial \Delta_0}_{|\Delta_0=0} = \sum_{j=1}^{m} A_j (H - s_j) \exp\left(\int_{s_j}^{H} r_{FW}(0, \tau)d\tau\right) = 0$$

where the derivative is taken at $\Delta_0 = 0$. Solving for H and multiplying and dividing by $P(0, H)$, we have the duration condition[16]:

$$D(A) = \frac{\sum_{j=1}^{m} A_j s_j \exp\left(-\int_0^{s_j} r_{FW}(0, \tau)d\tau\right)}{\sum_{j=1}^{m} A_j \exp\left(-\int_0^{s_j} r_{FW}(0, \tau)d\tau\right)} = H$$

$\square$

Proof of Theorem 2 Using the notation introduced in the proof of Theorem 1, define:

$$f(s) \equiv \exp\left(\int_s^H \Delta_0(\tau)d\tau\right)$$

so that: $f(H) = 1$, $f'(s) = -f(s)\Delta_0(s)$, $f''(s)\left[\Delta_0^2(s) - \Delta_0'(s)\right] \geq 0$ (by the "convexity condition")

Then the change in value is:

$$\Delta A_0(H) \equiv G(r_{FW} + \Delta_0) - G(r_{FW}) = G(r_{FW} + \Delta_0) = \sum_{j=1}^{m} a_j(f(s_j) - 1)$$

and by Taylor (exact) formula:

$$f(s_j) = f(H) + (s_j - H)f'(H) + \frac{1}{2}(s_j - H)^2 f''(\xi_j)$$

---

[16] Note that the second order (convexity) condition is always satisfied in the single liability case.

so that, substituting:

$$G(r_{FW} + \Delta_0) = -\sum_{j=1}^{m} a_j(s_j - H)\Delta_0(H) + \frac{1}{2}\sum_{j=1}^{m} a_j(s_j - H)^2 f(\xi_j)\left[\Delta_0^2(\xi_j) - \Delta_0'(\xi_j)\right]$$

$$= \frac{1}{2}\sum_{j=1}^{m} a_j(s_j - D)^2 f(\xi_j)\left[\Delta_0^2(\xi_j) - \Delta_0'(\xi_j)\right] \geq 0$$

(1)

where the second equality comes from the duration matching hypothesis D=H and the inequality from the "convexity condition". $\qquad\square$

An alternative proof is given by Montrucchio and Peccati [22].

Proof of Corollary of Theorem 2   Using Taylor's formula around H:

$$f(s_j) \simeq f(H) + (s_j - H)f'(H) + \frac{1}{2}(s_j - H)^2 f''(H)$$

$$= 1 - (s_j - H)\Delta_0(H) + \frac{1}{2}(s_j - H)^2\left[\Delta_0^2(H) - \Delta_0'(H)\right]$$

so that, for $H = D$:

$$\frac{\Delta A_0(H)}{A_0(H)} \equiv \frac{G(r_{FW} + \Delta_0)}{\bar{L}} \simeq \frac{1}{2\bar{L}}\sum_{j=1}^{m} a_j(s_j - D)^2\left[\Delta_0^2(D) - \Delta_0'(D)\right] \equiv \frac{1}{2}M_0^2(D)\left[\Delta_0^2(D) - \Delta_0'(D)\right]$$

$\qquad\square$

Proof of Theorem 3   From the first order approximation for $f(s) = e^{x(s)} \geq 1 + x(s)$ we have:

$$G(r_{FW} + \Delta_0) = \sum_{j=1}^{m} a_j(f(s_j) - 1) \geq \sum_{j=1}^{m} a_j \int_{s_j}^{H} \Delta_0(\tau)d\tau$$

By the integral formula:

$$\Delta_0(H) - \Delta_0(\tau) = \int_{\tau}^{H} \Delta_0'(u)du$$

so that, integrating and using the Fubini theorem to change the order of integration:

$$\int_{s_j}^{H} \Delta_0(\tau)d\tau = (H - s_j)\Delta_0(H) - \int_{s_j}^{H}\int_{\tau}^{H} \Delta_0'(u)dud\tau = (H - s_j)\Delta_0(H) - \int_{s_j}^{H} \Delta_0' \int_{s_j}^{u} d\tau du$$

$$= (H - s_j)\Delta_0(H) - \int_{s_j}^{H} \Delta_0'(u)(u - s_j)du$$

If $K_0 \equiv \max_u \{\Delta_0'(u)\}.,$

$$\int_{s_j}^{H} \Delta_0(\tau)d\tau \geq (H - s_j)\Delta_0(H) - K_0\int_{s_j}^{H}(u - s_j)du = (H - s_j)\Delta_0(H) - \frac{1}{2}K_0(H - s_j)^2$$

(2)

so that, using the duration matching condition $D = H$:

$$G(r_{FW} + \Delta_0) \geq -\frac{1}{2}K_0\sum_{j=1}^{m} a_j(D - s_j)^2 = -\frac{1}{2}K_0M_0^2(D)\bar{L}$$

$\square$

Shiu [27] uses an exact Taylor formula for $\int_{s_j}^{H} \Delta_0(\tau)d\tau$ and Nawalkha and Chambers [23] use higher-order approximations for $f(s)$.

Appendix 2: The shift function

The Fong–Vasicek [11] sufficient condition for immunization is a constraint on the shift function:

$$\Delta_t(\tau) \equiv r_{FW}(t + dt, \tau) - r_{FW}(t, \tau) :$$
$$\Delta_t^2(\tau) - \Delta_t'(\tau) = c \geq 0, \forall \tau$$

Note that the term "convexity condition" stems from the fact that under this condition the function:

$$f(s) \equiv \exp\left(\int_{s}^{H} \Delta_t(\tau)d\tau\right)$$

is convex ($f''(s) \geq 0$).

Omitting the subscript, we can check that $\Delta(\tau) = \text{const}$ (uniform additive positive or negative shift) satisfies the convexity condition. The same holds for any shift for which $\Delta'(\tau)$ is negative (decreasing term structure of shifts): for example $\Delta(\tau) = \exp(-a\tau)$ for $a = \sqrt{c} \geq 0$. In general, the convexity condition is an ordinary, nonlinear differential equation of the Riccati type.

Using the substitution $\Delta = -u'/u$ we obtain:

$$u''(\tau) - c(\tau)u(\tau) = 0$$

The special case $c(\tau) = \tau$ is well known because one solution is the Airy function (Abramowitz and Stegun (eds.) [1], ch. 10, p.446):

$$u(\tau) = \frac{\sqrt{\tau}}{3}\left[I_{-\frac{1}{3}}\left(\frac{2}{3}\tau^{3/2}\right) - I_{\frac{1}{3}}\left(\frac{2}{3}\tau^{3/2}\right) \right]$$

where $I_\nu(x)$ is the modified Bessel function of order ν (Abramowitz and Stegun (eds.) [1], ch. 9 p. 375).

The integral representation is:

$$u(\tau) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{x^3}{3} + x\tau\right)dx$$

so that:

$$\Delta(\tau) = \frac{\int_0^\infty \sin\left(\frac{x^3}{3} + x\tau\right)x\,dx}{\int_0^\infty \cos\left(\frac{x^3}{3} + x\tau\right)dx}$$

In the stochastic case, the shift function is the stochastic change in the forward rate $dr_{FW}(t, \tau)$. In the Vasicek term structure model it is given by $e^{-r\tau}dr(t)$, i.e. the change in the short term rate smoothed by a negative exponential. Using the shift function implied by the EIOPA Stress Test (see Fig. 2) we obtain the convexity condition displayed in Fig. 4.

Appendix 3: The stochastic approach to immunization

Under the no-arbitrage approach of financial markets, the case of "flat shifts" and the immunization property $A(H) \geq \bar{L}$ imply a riskless arbitrage and are therefore not compatible with equilibrium [3, 15]. In continuous time, under stochastic term structure models, the immunization condition means that the value of assets in the next instant
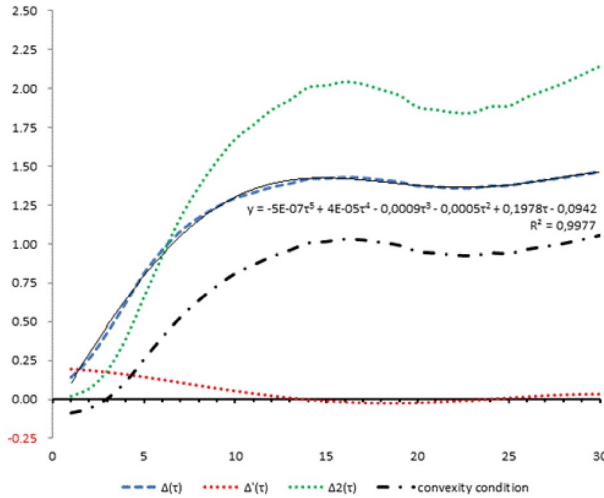
Fig. 4 EIOPA shift function $\Delta(\tau)$ and convexity condition

must be equal to the value of liabilities, so that, in differential terms: $dA(t) = dL(t)$ and immunization is guaranteed if the value of assets and liabilities have the same sensitivity to the state variables [2]. This has suggested the definition of a generalized or stochastic duration concept [4].

As an example of this stochastic approach to the asset-liability problem, let us consider the case of the extended-Vasicek [31], no arbitrage term structure model [14]:

$$dr(s) = [\hat{a}(s) - kr(s)]ds + \sigma d\hat{Z}(s)$$

where the time-dependent drift $\hat{a}(s)$ is obtained in order to guarantee a perfect fit of the theoretical term structure with the current yield curve:

$$\hat{a}(s) = kr_{FW}(t,s) + \frac{\partial r_{FW}(t,s)}{\partial s} + \frac{\sigma^2}{2k}(1 - e^{-2k(s-t)})$$

The solution for $r(s)$ is:

$$r(s) = r_{FW}(t,s) + \frac{\sigma^2}{2}G^2(t,s) + \sigma \int_t^s e^{-k(s-u)}d\hat{Z}(u)$$

$$G(t,s) = \frac{1 - e^{-k(s-t)}}{k}$$

and the relative difference between ex post and ex ante portfolio value is:

$$\frac{A(H) - A_0(H)}{A_0(H)} = \frac{\sum_{j=1}^{m} A_j \exp\left(\int_{s_j}^{H} r(u)du\right) - \sum_{j=1}^{m} A_j \exp\left(\int_{s_j}^{H} r_{FW}(0,\tau)d\tau\right)}{\sum_{j=1}^{m} A_j \exp\left(\int_{s_j}^{H} r_{FW}(0,\tau)d\tau\right)}$$

$$= \frac{\sum_{j=1}^{m} A_j \exp\left(\int_{s_j}^{H} r_{FW}(0,\tau)d\tau\right)\left[\exp\left(\int_{s_j}^{H} r(u)du - \int_{s_j}^{H} r_{FW}(0,\tau)d\tau\right) - 1\right]}{\sum_{j=1}^{m} A_j \exp\left(\int_{s_j}^{H} r_{FW}(0,\tau)d\tau\right)}$$

$$\approx \frac{\sum_{j=1}^{m} a_j\left[\int_{s_j}^{H}\left(r(u) - r_{FW}(0,u)\right)du\right]}{\sum_{j=1}^{m} a_j} \equiv \sum_{j=1}^{m} b_j X_{0j}$$

The variable X_{t_j} has an explicit form in this case:

$$X_{tj} = \int_{s_j}^{H}\left(r(u) - r_{FW}(t,u)\right)du = \frac{\sigma^2}{2k^2}\int_{s_j}^{H}\left(1 - e^{-k(u-t)}\right)^2 du + \sigma\int_{s_j}^{H}\int_{t}^{u} e^{-k(u-v)}d\hat{Z}(v)du$$

with mean:

$$E\left(X_{tj}\right) = \frac{\sigma^2}{2k^2}\int_{s_j}^{H}\left(1 - e^{-k(u-t)}\right)^2 du$$

$$= \frac{\sigma^2}{k^3}\left[\frac{k}{2}(H - s_j) + e^{-k(H-t)} - e^{-k(s_j-t)} - \frac{e^{-2k(H-t)}}{4} + \frac{e^{-2k(s_j-t)}}{4}\right]$$

and covariance

$$Cov\left(X_{ti}, X_{tj}\right) = \sigma^2 E\left[\int_{s_i}^{H}\int_{t}^{u} e^{-k(u-v)}d\hat{Z}(v)du \int_{s_j}^{H}\int_{t}^{s} e^{-k(s-v)}d\hat{Z}(v)ds\right]$$

$$= \sigma^2 \int_{s_i}^{H}\int_{s_j}^{H}\int_{t}^{\min(u,s)} e^{-k(u-v)-k(s-v)}dvdsdu$$

so that a risk-return approach can be analytically developed.

Appendix 4: Practical aspects of the computations of some financial variables

In this Appendix we give some technical details of the computations involving financial variables.

The computation of term structure changes

In the empirical application we use daily data and we calculate, from the 1 year-forward rates, $r_{FW}(t,\tau, \tau+1)$ for $\tau = 1,\ldots,25$ the daily changes:

$$\Delta_t(\tau) \equiv r_{FW}(t + 1, \tau, \tau + 1) - r_{FW}(t, \tau, \tau + 1)$$
$$\Delta'_t(\tau) \equiv \Delta_t(\tau + 1) - \Delta_t(\tau)$$
$$\Delta S_t(\tau) \equiv 1/2[\Delta_t^2(\tau) - \Delta'_t(\tau)]$$

The analytic, closed form computation of the portfolio cash-flow dispersion

It is possible to compute in closed-form, i.e. via an analytic formula, the cash-flow dispersion of a securities portfolio. Portfolio cash-flow dispersion turns out to be a quadratic function of portfolio duration, of its yield-to-maturity and its convexity[17]:

$$M^2(D) = C^{Mod} \cdot (1 + y)^2 - D \cdot (D + 1)$$

$M^2(D) =$ variance of the asset cash-flow maturities around their *duration*; $y =$ portfolio *yield-to-maturity*, $D =$ portfolio *duration* (average time-to-maturity), $C^{Mod}. =$ portfolio modified convexity.

The variance with respect to horizon H is obtained as:

$$M^2(H) = M^2(D) + (D - H)^2$$

Yield, duration and convexity aggregation at portfolio level

- The computation of the portfolio aggregate yield-to-maturity

Traditionally one computes the average portfolio yield, weighted by the amount invested (w.r.t. either book-value or market value). This estimate represents an approximation to the yield-to-maturity as seen from the valuation date. A better alternative seems to consist of computing an estimate that takes into account the different time distance of cash-flows from the valuation date: one weights each yield by the "dollar duration" of its security. One obtains the so called WADD yield (*Weighted Average Dollar Duration Yield,* see Grondin [12]). The Dollar Duration is the product of *duration* and price of a security.

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[17] See for example, Smith [30], p. 1670, formula (9) or de La Grandville [5], p. 163, formula (19). With continuous compounding, the formula simplifies into $M^2 = C - D$.

- The computation of average portfolio *Duration* and *Convexity*

The (modified) duration was downloaded from Bloomberg, for each security in the data set (at the date 31/12/2014). Portfolio duration is the average (weighted by market value amounts) of the durations of single securities.

Convexity was also downloaded from Bloomberg, for each security (at the date 31/12/2014). Bloomberg approximated the second derivative (of price w.r.t. yield) as the "central difference" by varying yield-to-maturity by ± 1 basis point, that is[18]:

$$C^{Mod} \cong \frac{1}{P} \cdot \frac{\frac{P^+ - P}{\Delta y} - \frac{P - P^-}{\Delta y}}{\Delta y} = \frac{1}{P} \cdot \underbrace{\frac{P^+ - 2 \cdot P + P^-}{(\Delta y)^2}}_{\Delta y = 0.01\%} = 10,000 \cdot \frac{P^+ - 2 \cdot P + P^-}{P}.$$

Convexity is a "quadratic" measure and hence it can be linearly aggregated (by summation): by weighting with the amounts (at market value) one obtains portfolio convexity.

- The bond benchmarks for maturity classes

The four bond classes used in the empirical application are represented by the following benchmark bonds:

Bond index	Bond ISIN	Time-to-maturity (years)	Duration (years)	Convexity	M-squared
3	IT0004867070	2.84	2.72	10.31	0.21
5	IT0003644769	5.09	4.58	27.08	1.54
10	IT0004513641	10.17	8.27	86.32	9.59
20	IT0003535157	19.60	13.46	241.60	46.84

- M-squared for a portfolio of bonds

Let $B_T = B_1 + B_2$ a portfolio of 2 bonds with cash flows c_{11}, c_{12} and c_{21}, c_{22} respectively at times t_{11}, t_{12} and t_{21}, t_{22} respectively.
Using a flat term structure:

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[18] Bloomberg data include in the price P accrued interest (full price = clean price + accrued interest) and assume a variation of ± 1 basis point w.r.t. the yield implicit in the price P.

$$B_1 = c_{11} \exp\left(-rt_{11}\right) + c_{12} \exp\left(-rt_{12}\right)$$

$$B_2 = c_{21} \exp\left(-rt_{21}\right) + c_{22} \exp\left(-rt_{22}\right)$$

$$D_1 = t_{11}c_{11} \exp\left(-rt_{11}\right)/B_1 + t_{12}c_{12} \exp\left(-rt_{12}\right)/B_1$$

$$D_2 = t_{21}c_{21} \exp\left(-rt_{21}\right)/B_2 + t_{22}c_{22} \exp\left(-rt_{22}\right)/B_2$$

$$D_T = t_{11}c_{11} \exp\left(-rt_{11}\right)/B_1\left(B_1/B_T\right) + t_{12}c_{12} \exp\left(-rt_{12}\right)/B_1$$

$$\left(B_1/B_T\right) + t_{21}c_{21} \exp\left(-rt_{21}\right)/B_2\left(B_2/B_T\right)$$

$$+ t_{22}c_{22} \exp\left(-rt_{22}\right)/B_2\left(B_2/B_T\right) = D_1 B_1/B_T + D_2 B_2/B_T$$

$$M_1^2 = \left(t_{11} - D_1\right)^2 c_{11} \exp\left(-rt_{11}\right)/B_1 + \left(t_{11} - D_1\right)^2 c_{12} \exp\left(-rt_{12}\right)/B_1$$

$$M_2^2 = \left(t_{21} - D_2\right)^2 c_{21} \exp\left(-rt_{21}\right)/B_2 + \left(t_{22} - D_2\right)^2 c_{22} \exp\left(-rt_{22}\right)/B_2$$

$$M_T^2 = \left(t_{11} - D_T\right)^2 c_{11} \exp\left(-rt_{11}\right)/B_1\left(B_1/B_T\right) + \left(t_{12} - D_T\right)^2 c_{12} \exp\left(-rt_{12}\right)/B_1\left(B_1/B_T\right)$$

$$+ \left(t_{21} - D_T\right)^2 c_{21} \exp\left(-rt_{21}\right)/B_2\left(B_2/B_T\right) + \left(t_{22} - D_T\right)^2 c_{22} \exp\left(-rt_{22}\right)/B_2\left(B_2/B_T\right)$$

$$= \left[M_1^2 + \left(D_1 - D_T\right)^2\right]\left(B_1/B_T\right) + \left[M_2^2 + \left(D_2 - D_T\right)^2\right]\left(B_2/B_T\right)$$

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