Tail Dependence Models for Risk Management

Fabrizio Durante

Dipartimento di Scienze dell'Economia
Università del Salento, Lecce, Italy
fabrizio.durante@unisalento.it
sites.google.com/site/fbdurante

IVASS, Rome, July 2017

The general framework

Given

- Risk (random) factors $\mathbf{X} = (X_1, \ldots, X_d) \sim F$, where

$$F(x_1, \ldots, x_d) = \mathbb{P}(X_1 \leq x_1, \ldots, X_d \leq x_d)$$

- A financial position $\psi(\mathbf{X})$
- A risk measure/pricing function: ρ

Our goal is to

$$\text{calculate } \rho(\psi(\mathbf{X}))$$

Warning: $\rho(\psi(\mathbf{X}))$ depends on the *joint* distribution function $F_{\mathbf{X}}$ of $\mathbf{X}$ and, especially, on its behavior in the *tails*.

Current practice

Given some risk factors $\mathbf{X} = (X_1, \dots, X_d)$, we proceed as follow:

- First, estimate the marginal behavior F_i of each X_i, i.e.

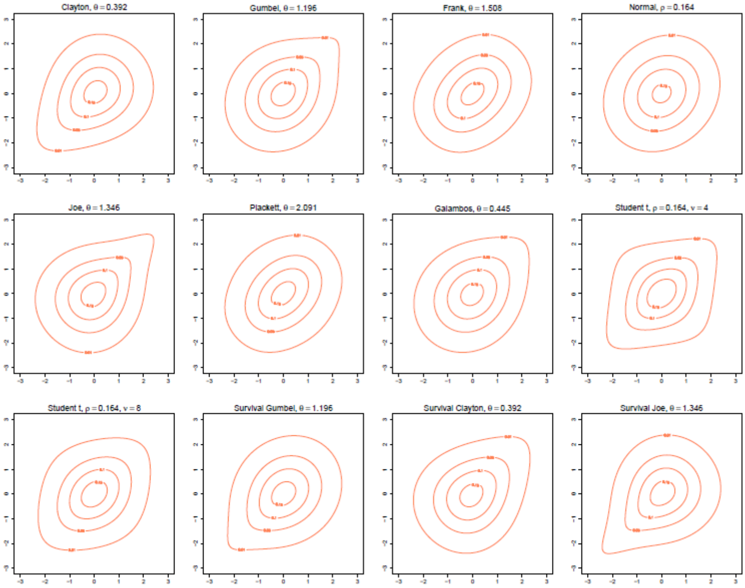
$$F_i(x) = \mathbb{P}(X_i \leq x).$$

- Find a copula C (i.e. a d.f. with uniform marginals) such that

$$\mathbf{X} \sim F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d)).$$

- Estimate $\rho(\psi(\mathbf{X}))$ either analytically or by means of a MC simulation from the probability d.f. F of $\mathbf{X}$.

> It is necessary to construct dependency models that reflect observed and expected dependencies without formalizing the structure of those dependencies with cause-effect models. The theory of copulas provides a comprehensive modelling tool that can reflect dependencies in a very flexible way.
>
> International Actuarial Association, 2004



Bivariate sample clouds from the d.f. $F = C(F_1, F_2)$ where $F_1, F_2 \sim N(0, 1)$, while C comes from different copula families.

Tail dependence coefficients

Let X and Y be continuous r.v.'s with d.f.'s F_X and F_Y, respectively. The upper tail dependence coefficient λ_U of (X, Y) is defined by

$$\lambda_U = \lim_{t \to 1^-} \mathbb{P}\left(Y > F_Y^{(-1)}(t) \mid X > F_X^{(-1)}(t)\right);$$

and the lower tail dependence coefficient λ_L of (X, Y) is defined by

$$\lambda_L = \lim_{t \to 0^+} \mathbb{P}\left(Y \le F_Y^{(-1)}(t) \mid X \le F_X^{(-1)}(t)\right);$$

provided that the above limits exist.

(Sibuya, 1960; Joe, 1993)

TDC's can be calculated from the copula C of (X, Y); in fact,

$$\lambda_L = \lim_{t \to 0^+} \frac{C(t,t)}{t} \quad \text{and} \quad \lambda_U = \lim_{t \to 1^-} \frac{1 - 2t + C(t,t)}{1 - t}.$$

Tail concentration function

An auxiliary function that may serve to visualize the tail dependence of a copula C is the so–called tail concentration function, defined as the function $q_C\colon (0,1) \to [0,1]$ given by

$$q_C(t) = q_L(t) \cdot \mathbf{1}_{(0,0.5]}(t) + q_U(t) \cdot \mathbf{1}_{(0.5,1)}(t),$$

where

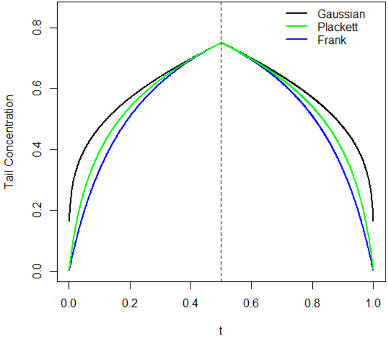
$$q_L(t) = \frac{C(t,t)}{t}, \qquad q_U(t) = \frac{1-2t+C(t,t)}{1-t}.$$

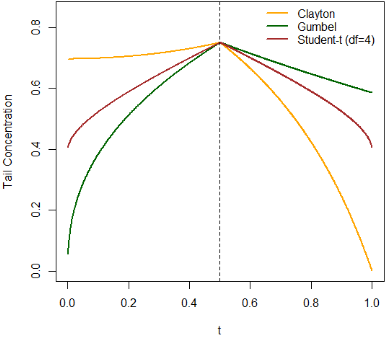
(Venter, 2001; D., Fernández-Sánchez and Pappadà, 2015)

Notice that:

- if C is the comonotonicity copula (positive monotone dependence), then $q_C = 1$;
- $q_C(0.5) = (1+\beta_C)/2$, where $\beta_C = 4C(0.5,0.5) - 1$ is the Blomqvist's measure of association related to C.

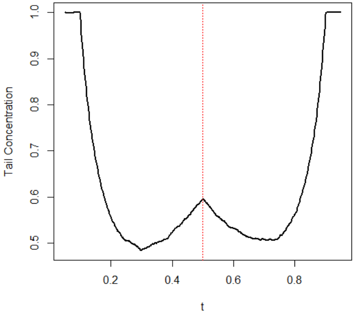
Tail concentration for popular families of copulas





Tail concentration function for various families of copulas with zero LTDC and UTDC (left) and with possibly non-zero LTDC or UTDC (right).

Tail concentration for patchwork copulas



Tail concentration functions for copulas obtained via patchwork methods.

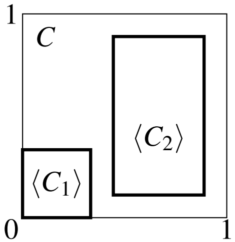
Outline

The main idea

A patchwork copula derived from a fixed copula C is any copula $\widetilde{C}$ such that:

$$\widetilde{C} = C \quad \text{on } [0,1]^d \setminus \cup_i B_i,$$

where each $B_i \subseteq [0,1]^d$ is a d–dimensional box in which the probability mass of $\widetilde{C}$ is distributed according to another copula C_i.



Applications:

- Modification of tail dependence behaviour
- Approximation of copulas

Patchwork copulas include ordinal sums, multilinear copula extensions, Bernstein copulas, gluing copulas, upper comonotonicity, etc.

Patchwork copulas

Let C and C_B be d–dimensional copulas and let $B = [\mathbf{a}, \mathbf{b}]$ be a non-empty box contained in $\mathbb{I}^d$ such that $\mathbb{P}_C(B) = \alpha > 0$. The function $C^* : \mathbb{I}^d \to \mathbb{I}$ given by

$$C^*(\mathbf{u}) = \mathbb{P}_C\left([\mathbf{0}, \mathbf{u}] \cap B^c\right) + \alpha\, C_B\left(\widetilde{F}_B^1(u_1), \dots, \widetilde{F}_B^d(u_d)\right)$$

is a copula, where, for every $x_i \in [a_i, b_i]$,

$$\widetilde{F}_B^i(x_i) = \frac{1}{\alpha}\mathbb{P}_C\left([a_1, b_1] \times \cdots \times [a_{i-1}, b_{i-1}] \times [a_i, x_i] \times [a_{i+1}, b_{i+1}] \times \cdots \times [a_d, b_d]\right).$$

The copula C^* is called patchwork of (B, C_B) into C and it is denoted by the symbol $C^* = \langle B, C_B \rangle^C$.

(D., Fernández–Sánchez and Sempi, 2013)

Patchwork copulas: simulation

Consider the patchwork $C^* = \langle B, C_B \rangle^C$, where $B = [\mathbf{a}, \mathbf{1}]$.

An algorithm for generating a random sample from C^* goes as follows.

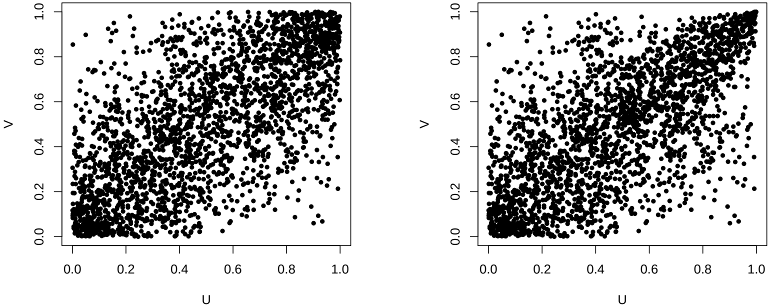
Algorithm

1. Generate $\mathbf{u}$ from the copula C.
2. If $\mathbf{u} \in B$, then
   1. Generate $\mathbf{v}$ from the copula C_B.
   2. For $i = 1, 2, \dots, d$ set $w_i = (\widetilde{F}_B^i)^{-1}(v_i)$.

   Otherwise, set $\mathbf{w} = \mathbf{u}$.
3. Return $\mathbf{w}$.

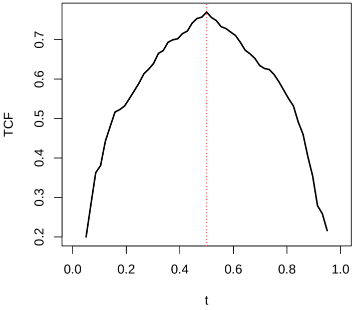
It can be used for “stress testing” the tail of the distribution.

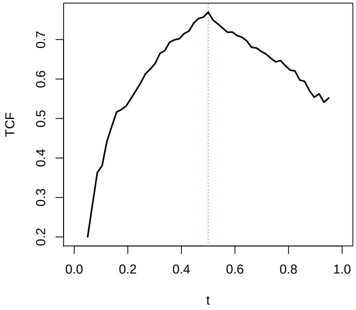
Patchwork copulas: simulation



Random sample of 2500 realizations from the Frank copula with $\tau = 0.50$ (left) the copula $\langle B, C_B \rangle^C$ where $B = [0.50, 1]^2$, C is the Frank copula with $\tau = 0.50$ and C_B is the Gumbel copula with $\tau = 0.50$.

Patchwork copulas: simulation





Tail concentration function from random sample of 2500 realizations from the Frank copula with $\tau = 0.50$ (left) the copula $\langle B, C_B \rangle^C$ where $B = [0.50, 1]^2$, C is the Frank copula with $\tau = 0.50$ and C_B is the Gumbel copula with $\tau = 0.50$.

Application: VaR and subadditivity

Consider two random losses L_1 and L_2 such that $L_1 = f(L_2)$ a.e. for some strictly increasing function f, i.e. they are comonotone and their copula is $M_2(u,v) = \min\{u,v\}$.
Then

$$\mathrm{VaR}_\alpha(L_1 + L_2) = \mathrm{VaR}_\alpha(L_1) + \mathrm{VaR}_\alpha(L_2).$$

However, it is not true that VaR_α is subadditive, i.e. for all losses L_1, L_2

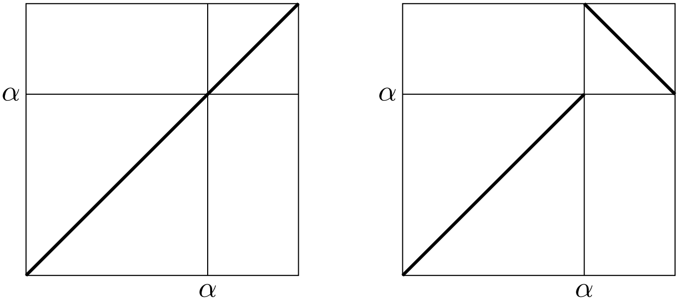
$$\mathrm{VaR}_\alpha(L_1 + L_2) \leq \mathrm{VaR}_\alpha(L_1) + \mathrm{VaR}_\alpha(L_2).$$

In fact,

$$\sup\{\mathrm{VaR}_\alpha(L_1 + L_2) \colon L_1, L_2 \text{ fixed }\} \geq \mathrm{VaR}_\alpha(L_1) + \mathrm{VaR}_\alpha(L_2).$$

Subadditivity reflects the idea that risk can be reduced by diversification.

Illustration: worst-case VaR copula for $d = 2$



Scatter plot from a comonotone copula (left) and from the copula giving the worst-case VaR (right). The copula for the right figure is based on works by Makarov (1981) and Rüschendorf (1982).

Illustration: worst-case VaR scenario

	$\tau = 1$	$\tau = 0.50$	$\tau = 0.00$	$\tau = -0.50$	$\tau = -1$
$\mathrm{VaR}_\alpha(L_1^{C^*}, L_2^{C^*})$	2.5631	2.5663	2.5749	3.0340	3.2897

Numerical approximation of $\mathrm{VaR}_{0.90}(L_1^{C^*}, L_2^{C^*})$ where $L_1, L_2, \sim N(0,1)$, $C^* = \langle [0.90, 1]^2, C_B \rangle^{M_2}$ for a Clayton copula C_B with Kendall's τ equal to the indicated value. Results based on 10^6 simulation from the given copula. The value corresponding to $\tau = -1$ is the worst-case VaR scenario. The value corresponding to $\tau = 1$ is the comonotonic scenario.

Outline

The goal

The aim is to ease the selection of the best copula that can be fitted to a random sample $(X_i, Y_i)_{i=1,\dots,n}$ starting from of a large number of possible parametric families. The "goodness criterion" depends on the tail dependence.

The procedure goes as follows:

- Consider a set of parametric copula models $\mathcal{C}^1, \dots, \mathcal{C}^k$ that may be appropriate for describing the unknown dependence structure in the given data, i.e. the so-called copula-test space.

(Michiels and De Schepper, 2008, 2013)

- Find of a (tail-dependence driven) 2D visualization of the copula-test space associated to a given dataset, and use it as the first step of model building.

(D., Fernández-Sánchez and Pappadà, 2015; Pappadà, D. and Torelli, 2017)

The Algorithm

Let $(X_i, Y_i)_{i=1,\dots,n}$ be a bivariate sample from an unknown copula.

1. Consider a set of k parametric copulas $C_1, C_2, \dots, C_k$ in the copula-test space that have been fitted to the available data.

2. Calculate a distance between the empirical copula C_n and C_i ($i = 1, \dots, k$) as

$$\sigma(C_n, C_i) = \int_a^b \left(q_{C_n}(t) - q_{C_i}(t)\right)^2 dt$$

- q_{C_n} is the *empirical* TCF given by

$$q_{C_n}(t) = \frac{C_n(t,t)}{t} \cdot \mathbf{1}_{(0,0.5)}(t) + \frac{1 - 2t + C_n(t,t)}{1-t} \cdot \mathbf{1}_{[0.5,1)}(t)$$

- q_{C_i} is the TCF associated with C_i.

The Algorithm

3. Calculate a distance between the i–th and the j–th copula in the copula test space via

$$\sigma(C_i, C_j) = \int_a^b \left(q_{C_i}(t) - q_{C_j}(t)\right)^2 dt$$

for $1 \leq i \neq j \leq k$.

4. Construct the distance matrix $\Delta = (\sigma_{ij})$, of order $(k+1)$, with elements

$$\begin{aligned} \sigma_{1j} =& \sigma(C_n, C_{j-1}), && j = 2, \dots, k+1 \\ \sigma_{ij} =& \sigma(C_{i-1}, C_{j-1}), && i,j = 2, \dots, k+1, \quad i < j \\ \sigma_{ii} =& 0, && i = 1, \dots, k+1 \end{aligned}$$

The Algorithm

5. Perform a non-metric scaling on Δ to find a low-dimensional map on which the inter-points distances d_{ij}'s are as close as possible to the original σ_{ij}'s
    - a monotonic transformation of the dissimilarities is calculated, which yields the *disparities* $\hat{d}_{ij}$, such that the $\hat{d}_{ij}$*s and the σ_{ij}'s have the same rank order
    - the optimum configuration is determined by minimising Kruskal's *stress*

$$s = \sqrt{\frac{\sum_{i<j}(\hat{d}_{ij} - d_{ij})^2}{\sum_{i<j} d_{ij}^2}}$$

6. Visualize the resulting set of $k+1$ points $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{k+1}$, in the q–dimensional Euclidean space, $q \in \{1, 2, \ldots, k\}$

Illustration: MSCI Index Data

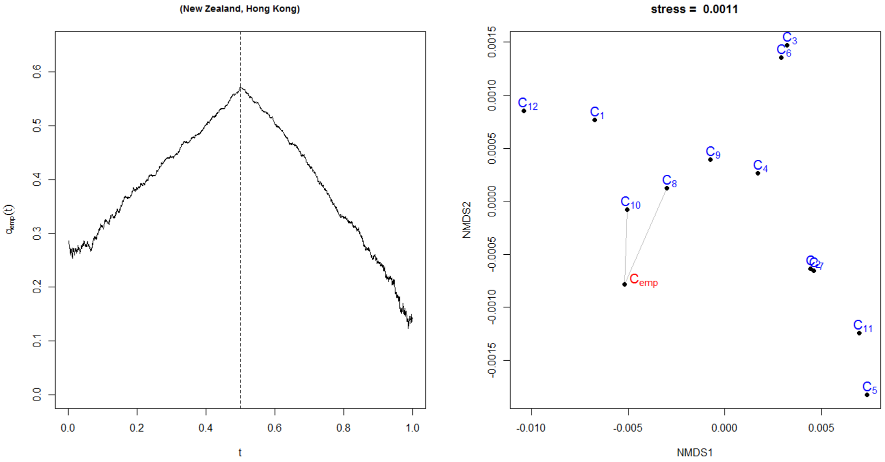
We use *Morgan Stanley Capital International* (MSCI) Developed Markets Index, which measures the equity market performance of 23 developed markets (daily observations from 2002-06-04 to 2010-06-10).

Preliminary steps:

- We fit a convenient ARMA-GARCH model to each time series to remove possible (conditional) mean and variance effects.
- We extract the pseudo–observations from the fitted residuals of the univariate time series focusing, hence, on the copula among the innovations of the series.
- We select a convenient copula-test space given by:

$\mathcal{C}^1$ = Clayton, *Archimedean*	$\mathcal{C}^2$ = Gumbel, *Archimedean* + *EV*
$\mathcal{C}^3$ = Frank, *Archimedean*	$\mathcal{C}^4$ = Normal
$\mathcal{C}^5$ = Joe, *Archimedean*	$\mathcal{C}^6$ = Plackett
$\mathcal{C}^7$ = Galambos, *EV*	$\mathcal{C}^8$ = Student's t, 4 df
$\mathcal{C}^9$ = Student's t , 8 df	$\mathcal{C}^{10}$ = Survival Gumbel
$\mathcal{C}^{11}$ = Survival Clayton	$\mathcal{C}^{12}$ = Survival Joe

Illustration: MSCI Index Data

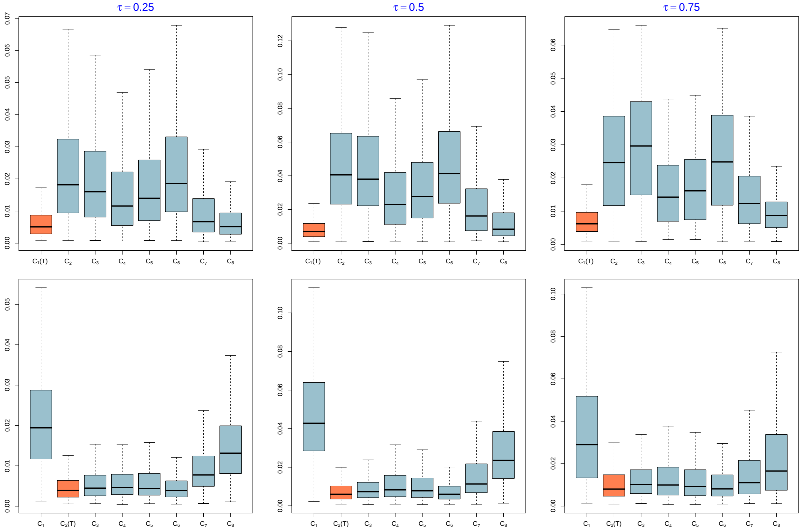


Empirical TCF (left) and two-dimensional representation of goodness-of-fit New Zealand–Hong Kong dependence structure, based on lower TCF (right). The estimated τ-value equals 0.1639.

Illustration: MSCI Index Data

Pairs plots for a three dimensional dataset of MSCI Indices. Lower panels display the values of Kendall' tau. Upper panels display the 2–dimensional TDC representation.

The Algorithm: finite-sample performances



Dissimilarities based on **lower TCF** between the empirical and the fitted copula–Clayton (C_1), Gumbel (C_2), Frank (C_3), Gaussian (C_4), Plackett (C_5), Galambos (C_6), Student-t, $\nu = 4$ (C_7), Surv. Gumbel (C_8)–when the "true" model is Clayton (in the first row) and Gumbel (in the second row), respectively. Sample size $n = 250$.

A related algorithm: Tail dependence-based clustering

In a similar spirit, an interesting way to visualize the relationships between observed variables is to determine clusters of homogeneous variables as a preliminary step of high–dimensional models (e.g., factor models, hierarchical models, etc.)

The main feature of this approach has three main steps:

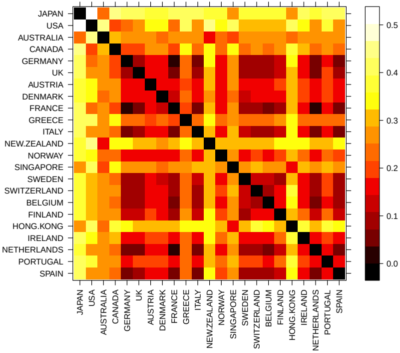
1. Determine a marginal distribution for each risk (in case of time series data).
2. Fix a copula-based measure of extreme dependence, which can be also estimated non-parametrically, for each pair of risks. For instance:

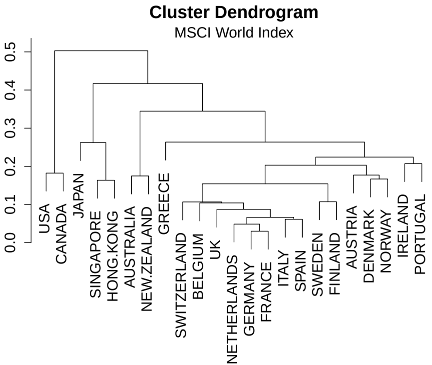
$$\sigma_{ij} = \|q_{\text{emp}}^{ij} - q_{M_2}\|,$$

   where q_{M_2} is the TCF of the comonotone copula M_2 and $\|\cdot\|$ any convenient norm.
3. Find a hierarchical structure with “bottom up” approach.

(D., Pappadà, Torelli, 2014; 2015)

Illustration: MSCI Index Data





Heat map matrix of dissimilarities (left) and dendrogram resulting from hierarchical clustering for the MSCI World Index Data according to complete linkage (right).

Concluding remarks

- We have presented the tail dependence coefficients and show some of their features via patchwork constructions.
- We have introduced a copula-based graphical tool to visualize the goodness-of-fit of a collection of parametric copula models at once.
- The given tool is based on a suitable measure of finite tail dependence in functional form, thus providing valuable indications for the choice of a copula model when the tail behaviour is of primary interest.
- A related method helps performing a cluster analysis of time series in order to detect groups of variables exhibiting higher association in the tails.

Questions? Comments?

Thanks for your attention!

- F. Durante and C. Sempi. *Principles of Copula Theory*. CRC/Chapman & Hall, Boca Raton, FL, 2016.
- F. Durante, J. Fernández-Sánchez, and C. Sempi. Multivariate patchwork copulas: a unified approach with applications to partial comonotonicity. *Insurance Math. Econom.*, 53(3):897–905, 2013.
- F. Durante, J. Fernández-Sánchez, and R. Pappadà. Copulas, diagonals and tail dependence. *Fuzzy Sets and Systems*, 264:22–41, 2015.
- F. Durante, R. Pappadà, and N. Torelli. Clustering of time series via non–parametric tail dependence estimation. *Statist. Papers*, 56(3):701–721, 2015.
- R. Pappadà, F. Durante, and N. Torelli. A graphical tool for copula selection based on tail dependence. In F. Mola, C. Conversano, and M. Vichi, editors, *Classification, (Big) Data Analysis and Statistical Learning*. Springer International Publishing, 2017.