Bonus–Malus systems with Weibull distributed claim severities

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Car Insurance Variables

- A priori variables: age, <u>gender</u>, type and use of car, country
- A posteriori variables: deductibles, <u>bonus-malus</u>

Bonus-Malus:

- Answer to **heterogeneity of behavior** of drivers
- Strongly influenced by **regulatory environment** and culture

EU rules on gender-neutral pricing in insurance.

From 21 December 2012, insurance companies in the European Union have to charge the same price to men and women for the same insurance products, without distinction on the grounds of gender.

Laurianne Krid, policy manager at FIA

"Women are safer drivers statistically, but they should pay according to their real risk, which can be calculated objectively."

"We want insurance to be based on criteria like type of vehicle, the age of the driver, how much you drive during the year, and **how many accidents you have had**."

From "E.U. Court to Insurers: Stop Making Men Pay More", By Leo Cendrowicz, TIME, Mar. 02, 2011

Today's talk

Advocate taking severity of claims into account in a BMS

- optimal BMS based both on the number of accidents of each policyholder and on the size of loss (severity) for each accident incurred.
- optimality is obtained by minimizing the insurer's risk.

Literature Review

- Categorisation of Claim Severities (Discrete)
  - Picard (1976): Large and Small
  - Lemaire (1995): Property Damage and Bodily Injuries
  - Pitrebois et al. (2006): Four types, Dirichlet Distribution
- Distributions of Claim Severities (Continuous)
  - Frangos and Vrontos (2001): Pareto distribtution
  - Valdez and Frees (2005): Burr XII long-tailed distribution
  - Ni, C., Pantelous (2014): Weibull distribution
  - Ni, Li, C., Pantelous (2014): Weibull and Pareto distribution

Lemaire (1995)

$$Premium = E(Frequency)\,\underbrace{E(Severity)}_{constant}$$

- optimal Bonus-Malus Systems (BMS) -assign to each policyholder a premium based only on the number of his accidents.
- same penalty for an accident of a small size or big size.
- optimality is obtained by minimizing the insurer's risk.

Example - Lemaire (2017)

On a third party liability insurance:

Number of claims	Observed policies	Poisson fit	NB fit
0	96, 978	96, 689.6	96, 985.5
1	9, 240	9, 773.5	9, 222.5
2	704	493.9	711.7
3	43	16.6	50.7
4	9	0.4	3.6
5+	0	0	0
Total	106, 974	106, 974	106, 974

Note: Poisson $Mean = 0.1011$ and $Variance = 0.1070$

Jean Lemaire, 2017, Easter School, Liverpool

Mixed Poisson distributions

- Obviously Poisson is not the best fit!
- Need a distribution that exhibits positive contagion (dependence)
- Still assume that each individual has claims according to a Poisson(λ) process
- However, assume λ is a continuous random variable with density $g(\lambda)$,

$$P(N(t)=n)=\int_0^\infty P(N(t)=\lambda \mid \lambda)g(\lambda)d\lambda$$

Negative Binomial (NB) Claim Frequency

Mixing Poisson with Gamma(α, τ) results in Negative Binomial

$$P(N = n) = \int_0^\infty \frac{\mathrm{e}^{-\lambda}\lambda^n}{n!} \cdot \frac{\lambda^{\alpha-1}\tau^{\theta}\mathrm{e}^{-\tau\lambda}}{\Gamma(\alpha)} \mathrm{d}\lambda = \binom{n+\alpha-1}{n} \tau^{\alpha} \left(\frac{1}{1+\tau}\right)^{\alpha+n}.$$

Bayesian Approach - Posterior Distribution $Gamma(\alpha + K, \tau + t)$

$$\mu(\lambda | k_1, k_2, \ldots, k_t) = \frac{(\tau + t)^{K+\alpha}\lambda^{K+\alpha-1}\mathrm{e}^{-(t+\tau)\lambda}}{\Gamma(\alpha + K)}, \quad K = \sum_{i=1}^{t} k_i$$

Best Estimate - Posterior Mean

$$E[Frequency] = \lambda_{t+1}(k_1, k_2, \ldots, k_t) = \frac{\alpha + K}{\tau + t}.$$

Average number of claims

- Apriori - *Gamma*(α, τ): $\hat{\lambda} = \frac{\alpha}{\tau}$
- Observe claim history: $\{k_1, k_2, \ldots, k_t\}$, $K = k_1 + \cdots + k_t$
- Aposteriori - *Gamma*$(\alpha + K, \tau + t)$: $\hat{\lambda} = \frac{\alpha+K}{\tau+t}$

Net Premium in Optimal BMS with NB

$$Premium = \underbrace{E(Frequency)}_{=\frac{\alpha+K}{\tau+t}} * \underbrace{E(Severity)}_{=constant}$$

Distributions for claims severity

Tail behaviours of three comparative distributions [Boland 2007]

$$
\begin{aligned}
Exponential : P(X > x) &= exp(-\theta x); \\
Weibull : P(X > x) &= exp(-\theta x^{\gamma}); \\
Pareto : P(X > x) &= \left(\frac{\theta}{\theta + x}\right)^{s}.
\end{aligned}
$$

Pareto Claim Severity

Mixing exponential with Inv.Gamma(m,s) results in Pareto(s,m)

$$F(x) = \int_0^\infty (1 - \mathrm{e}^{-\theta x}) \frac{e^{-m\theta}(\theta m)^{s+1}}{m\Gamma(s)} \, d\theta = 1 - \left(\frac{m}{m+x}\right)^s$$

Bayesian Approach - Posterior Distribution

$$\pi(\theta | \underbrace{x_1, x_2, \ldots, x_K}_{claims' \, history}) \quad \sim \quad Inv.Gamma(m+M, s+K), \quad M = \sum_{k=1}^{K} x_k$$

Best Estimate - Posterior Mean

$$E[Severity] = \frac{m+M}{s+K-1}, \quad M = \sum_{k=1}^{K} x_k$$

Net Premium in optimal BMS with NB and Pareto

$$Premium = \underbrace{\frac{\alpha + K}{\tau + t}}_{E(frequency)} \underbrace{\frac{m + M}{s + K - 1}}_{E(severity)}$$

Weibull Claim Severity

Mixing exponential with a Levy distribution

$$F(x) = \int_0^{\infty} (1 - \mathrm{e}^{-\theta x}) \frac{c}{2\sqrt{\pi\theta^3}} exp\left(-\frac{c^2}{4\theta}\right) \mathrm{d}\theta = 1 - exp\left(-c\sqrt{x}\right).$$

Bayesian Approach - Posterior Distribution

$$\pi(\theta | x_1, x_2, \ldots, x_K) = \frac{\left(\frac{\alpha'}{\beta'}\right)^{\frac{v}{2}} \theta^{v-1} exp\left(-\frac{1}{2}\left(\alpha'\theta + \frac{\beta'}{\theta}\right)\right)}{2B_v\left(\sqrt{\alpha'\beta'}\right)},$$

B_v modified Bessel function, $M = \sum_{k=1}^{K} x_k$, $\alpha' = 2M, \beta' = \frac{c^2}{2}, v = K - \frac{1}{2}$.

Best Estimate - Posterior Mean

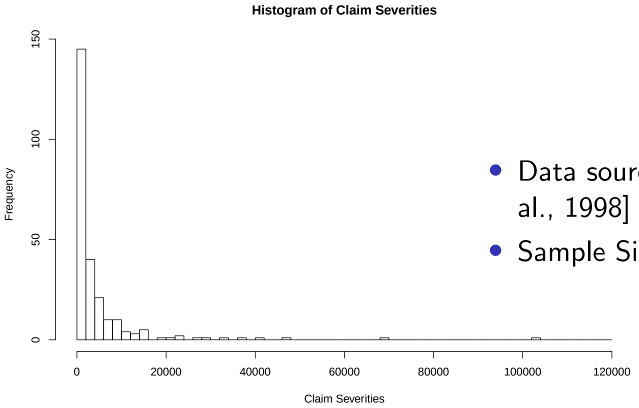
$$E[Severity] = \frac{2\sqrt{M}}{c} \frac{B_{K-\frac{1}{2}}(c\sqrt{M})}{B_{K+\frac{1}{2}}(c\sqrt{M})}.$$

Net Premium in optimal BMS with NB and Weibull

$$Premium = \frac{\alpha + K}{t + \tau} \cdot \left(\frac{2\sqrt{M}}{c} \frac{B_{K-\frac{1}{2}}(c\sqrt{M})}{B_{K+\frac{1}{2}}(c\sqrt{M})} \right),$$

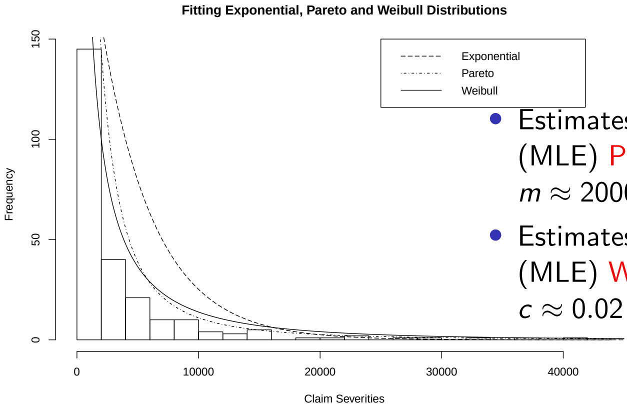
for $M > 0$, where $M = \sum_{k=1}^{K} x_k$.

Numerical Illustration



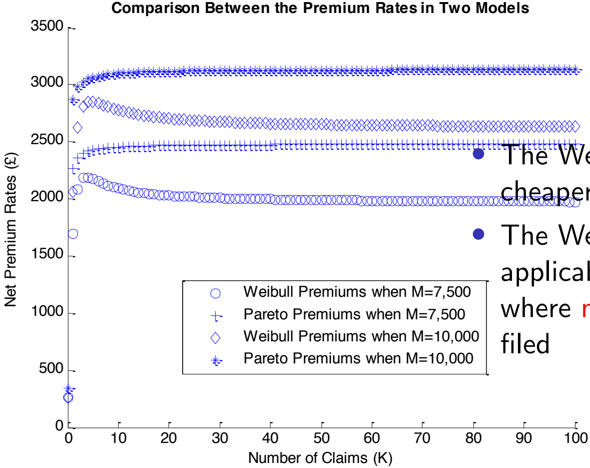
- Data source: [Klugman et al., 1998]
- Sample Size: 250

Fitting the Distributions



- Estimates of parameters (MLE) Pareto distribution: $m \approx 2000; s \approx 1.34;$
- Estimates of the parameter (MLE) Weibull distribution: $c \approx 0.02$

Analysis of the behaviour



- The Weibull model offers cheaper premium rates
- The Weibull model is more applicable on the scenario where many small claims are filed

Mixed Strategy

A mixture of the previous two models:

- $X \sim X_{Wei}$ when $X \leq z$
- $X \sim X_{Par}$ when $X > z$.

$$Premium = E_p[X_{Wei}]E_p[N_{Wei}](1-q) + E_p[X_{par}]E_p[N_{Par}]q.$$

with q denoting the probability that a claim cost exceeds a certain threshold z. Note that q and z can both be observed from a sample and E_p stands for the posterior mean.

Frequency Distribution

Suppose the total claim frequency is Negative Binomial distributed $N \sim NB(\alpha, \tau)$.

- Number of claims above the limiting amount z follows a Negative Binomial distribution, $N_{Par} \sim NB(\alpha, \tau q)$.
- Similarly $N_{Wei} \sim NB(\alpha, \tau(1-q))$.

Apriori: the means of claim frequency (Pareto and Weibull claims)

$$\begin{aligned} E[N_{Par}] &= \frac{\alpha}{\tau q}, \\ E[N_{Wei}] &= \frac{\alpha}{\tau(1-q)}. \end{aligned}$$

Aposteriori: the means

$$\begin{aligned} E_p[N_{Par}] &= \frac{\alpha + qK}{\tau q + t}, \\ E_p[N_{Wei}] &= \frac{\alpha + (1-q)K}{\tau(1-q) + t}. \end{aligned}$$

The Net Premium Formula

Premium $=$

$$\frac{\alpha+K(1-q)}{\tau(1-q)+t}\cdot\frac{2\sqrt{M_1}}{c}\frac{B_{K(1-q)-\frac{1}{2}}(c\sqrt{M_1})}{B_{K(1-q)+\frac{1}{2}}(c\sqrt{M_1})}(1-q)+\frac{\alpha+Kq}{\tau q+t}\cdot\frac{m+M_2}{s+Kq-1}q$$

- Kq or $K(1-q)$ are not necessarily integers.

Conclusions

Advantages

- Fair - as a result of Bayes rule
- Financially balanced - the average income of the insurer stays the same every year

Disadvantages

- high penalties $=>$ encourages uninsured driving; hit and run behaviour; change of insurer

Thus

- Instead of NB, Markov chains are used in practice.
- Similarly solution for Weibull/Pareto claims.

References

1. Ni, Constantinescu, Pantelous (2014). Bonus–Malus systems with Weibull distributed claim severities. *Annals of Actuarial Science* 8(02), 217–233.
2. Ni, Li, Constantinescu, Pantelous (2014): Bonus–Malus systems with hybrid claim severity distributions. *Vulnerability, Uncertainty and Risk*. American Society of Civil Engineers (online).

THANK YOU FOR YOUR ATTENTION!

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