

Quaderno n. 7

**Duration, convexity and the optimal management of
bond portfolios for insurance companies**

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February 2017

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DURATION, CONVEXITY AND THE OPTIMAL MANAGEMENT OF BOND PORTFOLIOS FOR INSURANCE COMPANIES

Riccardo Cesari (°) and Vieri Mosco (°°)

ABSTRACT

Two Fong-Vasicek immunization results are discussed and applied in relation to asset portfolios of a sample of Italian insurance companies managing life insurance with-profit savings. Firstly, we analyzed the contribution of Fong and Vasicek (1984) providing a lower bound on the “shortfall” of an asset portfolio, managed with a duration-matching target, in the face of an arbitrary shock to the term structure of interest rates. A “passive” management strategy emerges, aimed at minimizing risk, such that the exposure to an arbitrary variation of the shape of the term structure is minimized with respect to a risk measure that is increasing with the cash-flows dispersion. Secondly, Fong and Vasicek (1983) approach is generalized, in line with the classical risk-return approach to portfolio management, in a model which overturns the “passive” perspective of minimum risk exposure and looks for only a partial risk minimization in exchange for more return potential. The empirical application shows that such a perspective may be proved useful to highlight which segregated funds can be re-positioned over the efficient frontier, at the chosen level of the firm’s risk appetite.

JEL CLASSIFICATION: G22, G11, G12

KEY WORDS: Insurance companies, immunization, Fong-Vasicek theorem, bond portfolios, efficient frontier

ACKNOWLEDGEMENTS

We thank Oldrich Vasicek, Enzo Orsingher, Giovanni Rago, Fabio Polimanti, Stefano Pasqualini, Riccardo Poli, Antonio De Pascalis, Lino Matarazzo and the participants to a seminar of IVASS and Bank of Italy for their comments. All remaining errors are ours.

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1 Introduction

There are well known differences between the insurance and the banking business models. While traditional commercial banks rely upon maturity transformation (short term cash flow and medium-long term investments), insurance companies sell guarantees through a maturity matching between assets and liabilities.

This matching and the general approach known as integrated management of assets and liabilities (ALM) has essentially the aim to cope with the interest rate risk, i.e. the risk of an asymmetric impact of interest rate movements to the asset and the liability side of the balance sheet.

As an alternative to the basic cash flow matching solution, the duration concept, as suggested by Redington (1952), provided a feasible, more sophisticated but more vulnerable solution. In his seminal work, he introduced the term of immunization to indicate a balance sheet not exposed to interest rate movements. "Hedging" would be the modern equivalent¹.

The essential aspect is in the link between duration D and the price sensitivity to interest rates:

$$D = -\frac{1}{P} \frac{\partial P}{\partial r}$$

Considering the "equity duration" (in the broad sense of Surplus or Net Asset Value or Risk Capital: $E = A - L$), as an equilibrium indicator for the Equity position of a company against an overall variation in interest rates (a variation of their general, average level) we have:

$$D_E = D_A \cdot \frac{A}{A-L} - D_L \cdot \frac{L}{A-L} \geq 0.$$

Equivalently, the equity duration can also be regarded as the duration of liability plus the product of leverage and mismatch², in formulae $D_E = D_L + \frac{A}{E} \cdot (D_A - D_L)$.

Hence, given a duration mismatch, the higher the leverage the greater the sensitivity of equity to interest rates; or, in the other way round, given an asset leverage, the greater the duration mismatch the riskier is the value of equity in the face of interest rate changes.

However, even if insurance companies are usually not particularly vulnerable to liquidity risk as long as the assured capital is not callable (or callable under severe surrender penalties), they daily face the problem of finding high investment returns for a given set of liabilities.

In this respect, immunization, as a minimum (zero) risk approach, must be considered as a special case into the broader area of portfolio optimization, in which risk-return considerations provide an entire menu of management decisions, according to the firm's propensity to risk (risk appetite) and financial market opportunities. As in Markowitz (1959)

¹ See Appendix A for a review of the main results of the classical theory of immunization.

² See Messmore (1990) who shows that if $E > 0$ (the present value of assets greater than the present value of liabilities) the immunization of E , i.e. $D_E = 0$, implies $D_A < D_L$. In the following we assume a fixed income portfolio with $A = L$.

famous portfolio approach, the efficient frontier presents different combinations of risk and reward among which no single solution can be considered efficient, independently of the firm's risk aversion.

Fong and Vasicek (1983) introduced this risk-return, double dimension into the immunization literature³. Following their approach, we shall provide an implementable model according to which an insurance company could estimate the efficient frontier facing its fixed income balance sheet and evaluate, against arbitrary interest rate movements, how distant its portfolio is from efficiency and how much risk it is bearing with the chosen asset-liability mix.

In the next paragraph we shall consider the term structure movements as suggested by recent European stress tests and financial market dynamics. Then we analyze passive and active strategies of portfolio immunization, concluding the analysis with a recap of results and possible extensions.

2 The movements of the term structure of interest rates

It is well known that the movements of the term structure can be approximately decomposed in three basic elements: changes in the level, the slope and the curvature. The level consists of the mean interest rate across maturities ("average rate"), reducing the curve to a flat structure. The slope gives an increasing (decreasing) pattern to the curve and might be regarded as the differential between long and short rates ("term spread"). The curvature results in a humped or convex curve and takes into account asymmetric variations among short and long interest rates on the one side and medium-term rates on the other ("butterfly spread"). As shown in many empirical analyses, in "normal times" the level factor explains most of the historical variability of the interest rates, while the slope explains a residual amount and convexity only marginal variability⁴.

Depending on their asset and liability composition, insurance companies will be differently affected by future levels and shapes of interest rates.

The duration mismatch gives an approximate measure of this exposure: as shown in Fig. 1, country averages in Europe are widely dispersed, often far away from the perfect matching line⁵. This implies that market movements could impair the firms by different amounts, depending also on the different dynamics that will take place in real markets.

However, as we shall see, even well matched countries, like Italy, Great Britain, Belgium, Spain and Portugal, are not immunized from general movements in the shape of the term structure.

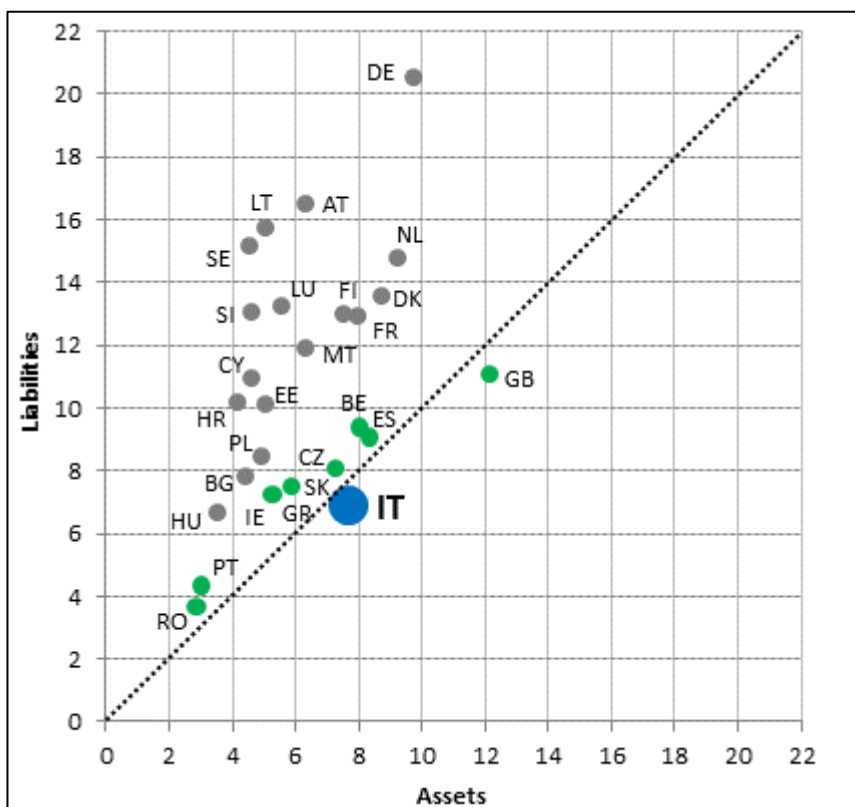
³ All Vasicek's papers have been recently collected in Vasicek (2016)

⁴ See Veronesi (2010), Ch. 4 for a basic treatment of this approach (Principal Component Analysis). In his estimates based on US data, the level component explains between 75% and 61%, of the variability, the slope component between 22% and 20% and the curvature component between 1% and 16%, depending on the interest rate maturity (between 3 months and 10 years).

⁵ The EIOPA calculation of the liability side in Fig. 1 does not take into account all the contract optionality available in different countries so that the country relative positions could be altered.

For example, in its 2014 stress test exercises, EIOPA, the European Authority for Insurance and Pension Funds, paid attention to the most dangerous scenario of lowering rates⁶.

FIG. 1 Duration of assets and liabilities in the EIOPA Stress Test Report



The two considered low yield scenarios were given by the curve of the so called “Japanese Scenario” (blue line in Fig. 2), i.e. a scenario based on a long-lasting flattening view of the term structure and by the curve of the so called “Inverse Scenario” (not reported here).

In Fig. 2 the two market term structure at the end of 2013 (green line or “Baseline scenario”) and at the end of 2014 (red line) are also given. Clearly, the “stressed” hypothesis has been overridden by the actual reality: the historical 2014 data (red line) can be regarded as a sort of “continuation” of the “Japanese scenario”.

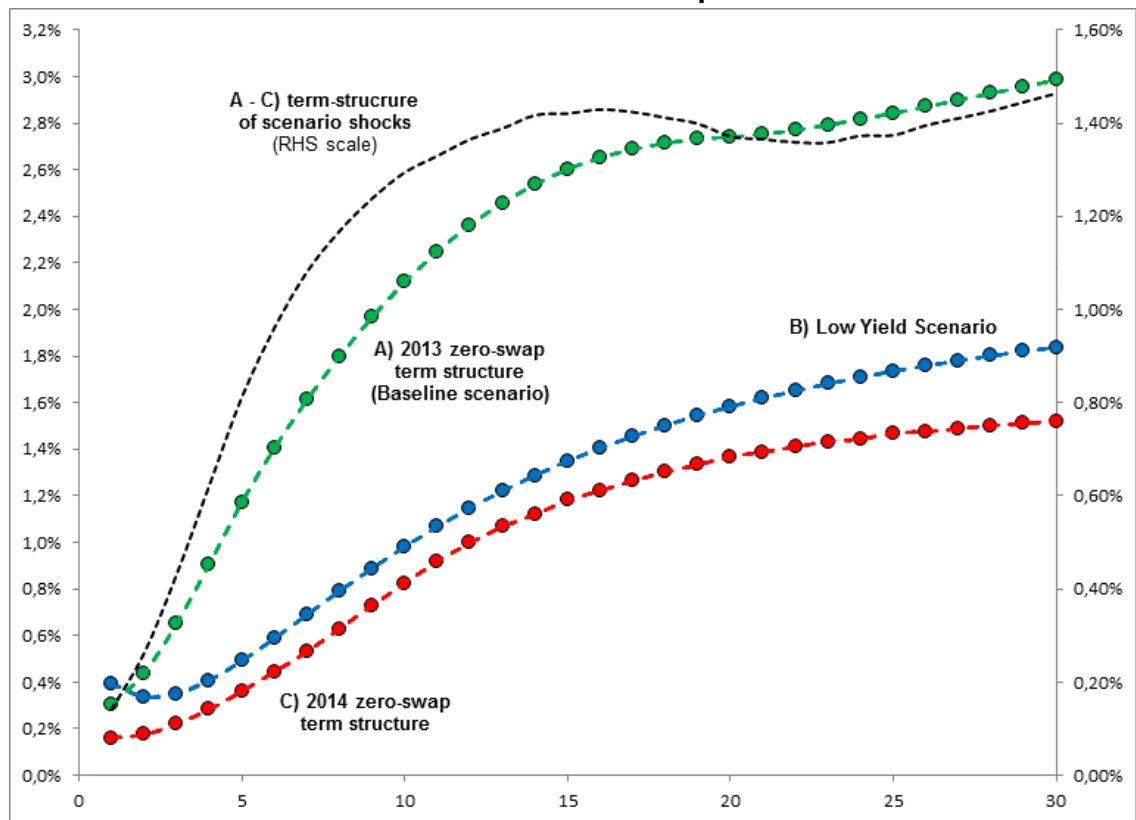
As can be seen, the hypothetical and the actual market movements are far from being a constant downsizing of the interest rates: the term structure of the shocks $\{\Delta(\tau), \tau = 1, 2, \dots, 30\}$ is not flat across maturities, and, moreover, it is mostly quite far from the typical 100 basis points assumption, ranging roughly from +20 bps to -120 bps.

⁶ For instance, see the Eiopa official presentation of the 2014 Stress Test in EIOPA (2014 a), slide 10.

According to the EIOPA 2014 Stress Test Report (EIOPA, 2014 b), the results of the stress exercise have been evaluated with respect to <<the size of duration mismatches between assets and liabilities as well as mismatches in internal rate of return of assets and liabilities>> which are <<considered the main drivers for the severity of an interest rate stress>>⁷.

However, following the analysis here proposed, in case of the asymmetric shocks to the term structure, the classical “duration gap” analysis should be augmented taking into account also portfolio convexity and its effect to the balance sheet reaction to interest rate changes.

FIG. 2 EIOPA 2014 Stress Test Scenarios for European interest rates



3 Passive strategy: minimizing the “immunization risk”

According to Fong and Vasicek (1983 a), the investment horizon (holding period) might be considered a strategic horizon with respect to which to guarantee some target return, \bar{R}_0 fixed ex-ante at time 0.

A lower bound for the ex post return is obtained by showing that the ex post portfolio value has a minimum percentage change (maximum shortfall) given by⁸:

⁷ See EIOPA (2014 b), section C, “Low Yield Module Description and Results”, paragraph 2, sub-paragraph 56.

⁸ See the Appendix A, Theorem 3 for a formal treatment.

$$\frac{\Delta A_0(H)}{A_0(H)} \geq -\frac{1}{2} \underbrace{M_0^2}_{\text{endogenous}} \cdot \underbrace{\max_{\tau}\{\Delta'_0(\tau)\}}_{\text{exogenous}}$$

where M_0^2 is a variance of times to payment and it can be regarded as a risk measure, specifically a risk measure for the imperfect immunization provided by the duration-matching strategy. By definition:

$$M_0^2 = \sum_{j=1}^m (s_j - D)^2 \cdot w_j,$$

where D is the duration and the weights w_j are given by the present values of the cash flows at time s_j .

Note that $\max\{\Delta'_0\}$ is the maximum change of the slope of the current term structure across its maturities.

In practice, the variation of the one-year forward rates $r_{FW}(0, \tau, \tau + 1)$ across the maturities is denoted by $\Delta_0(\tau)$ and its maximum marginal change is taken as finite:

$$\max \left\{ \frac{d}{d\tau} \Delta_0(\tau) \right\} = K_0 = \text{cost}$$

Note that, according to the Fong and Vasicek result, M^2 is a risk measure in that it captures the exposure to any arbitrary movement of the discount curve. It suffices, therefore, to minimize M^2 in order to reduce such exposure and this prudential strategy might be regarded as a “passive” management.

As duration represents a time average, M^2 is a time variance and can be expressed in terms of duration and convexity. Fong and Vasicek (1984) help us to understand it as a proper risk measure.

Let us consider the case of two portfolios, the “bullet” and the “barbell”.

A perfect “bullet” portfolio with maturity D has a minimum M^2 of zero. In practice, it is composed by low-coupon securities with maturities close to D so that M^2 is close to zero. Viceversa, a “barbell” portfolio is a set of very short and very long securities with large M^2 even if it has the same duration D .

These two portfolios with equal duration D are differently affected by a positive twist of interest rates (steepening) – i.e. an asymmetric variation of interest rates at the short and the long end of the curve, producing a decrease in short rates and an increase in long rates.

In fact, this twist has effects both in terms of “reinvested income” and “capital gain” because short term rates have become lower, producing lower coupon (higher cost prices) from reinvested income whilst long term rates are now higher, producing higher capital losses realized at the horizon. This will produce a shortfall of the portfolio value with respect to the target value (a negative twist will produce the opposite). But the two portfolios are differently (negatively) affected: the barbell portfolio (with higher M^2) is penalized twice, because, due to the barbell profile, i) a greater number of securities/cash flows reach sooner the maturity date with reinvestment at lower interest rates (due to the downward shock on

short interest rates) and ii) a higher and longer portion of the portfolio will be outstanding after the horizon date, producing a higher capital loss, given that securities with longer maturities will depreciate more heavily.

This means that both M^2 and the risk of the barbell portfolio are greater than those of the bullet.

Fixed coupon bonds are clearly affected by these effects and insurance firms, being a traditional investor for this kind of securities, cannot overlook the convexity of their portfolios.

In Italy, for example, 56 life insurance companies total 429 billion euros of segregated funds⁹ (at market values of December 2014) and more than 50% (218 billion euros) are invested fixed coupon bonds.

Using a sample of 13 medium-large life insurance companies (covering a market share of 75% in terms of collected premiums) we have applied Fong and Vasicek (1983) lower bound to the fixed coupon¹⁰, Government Bond (BTP) portfolios of Italian with-profit segregated funds.

In this sample, the duration gap, as of end 2013, is contained within a band of +/- 2 years (Fig. 3), so that the perfect matching assumption can be considered as a first approximation and we can concentrate on the second-order, convexity effects.

In order to calculate the portfolio sensitivity, the interest rate shock is assumed as an (instantaneous, time 0) reverse shift from the 2014 term structure back to the 2013 higher shape, i.e. it is assumed that the 2014 year end term structure instantaneously shifts back to the values prevailing a year before, performing a positive twist of the curve.

⁹ Italian segregated funds represent with-profit policies ruled under IVASS Regulation n. 38, limiting the alienation of securities from the fund only to selling operations. In particular, (see art. 9, co. 3), securities cannot be transferred to other funds of the company and must be held to maturity. The Italian profit participation mechanism works as follows.

For single premium policies the insured capital is

$$C(0) = \frac{V_0(1 + Tr)^n}{1 + w(1 + Tr)^n},$$

$$C(t) = C(t - 1)(1 + \rho(t)),$$

$$\rho(t) = \frac{\max(qR(t), K) - Tr}{1 + Tr},$$

where V_0 is the paid premium, Tr is the technical rate, n is the policy maturity, w is the loading rate, $R(t)$ is the year t gross return of the segregated fund, q is the participation rate, K is the guaranteed minimum rate. Note that the initial insured capital is calculated at the technical rate. The policy represents a participation mechanism where, each year, profit sharing cannot falls short of the guaranteed interest rate K , and the policyholder's account (technical reserve) is credited by a quota q of the investment profit $R(t)$. In practice, the mechanism gives rise to an embedded option (cliquet option), where the extra yield $qR(t)$, if greater than K , is annually "locked-in".

For annual premium policies, the insured capital is updated not in full but in proportion:

$$C(t) = C(t - 1)(1 + \rho(t)) - C(0)\rho(t)\frac{n - t}{n}.$$

Because of the profit participation mechanism, the liability cash-flows partly depend on the asset cash-flows, through the returns realized by the bonds in the segregated funds. Under low and stable interest rates, this effect is negligible.

¹⁰ For the sake of simplicity we applied the analysis only to fixed coupon bonds, even if floating or indexed coupons and callable bonds could be included as well.

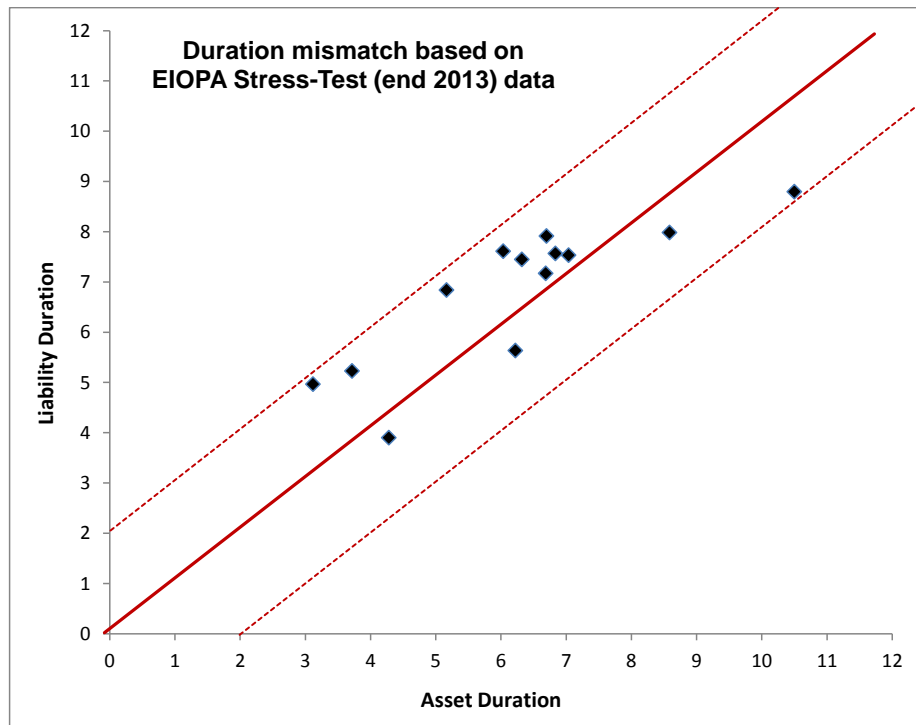
The variation of forward rates around the Italian market duration (about 7 years) is 10 bps:

$$K_0 = \max_{\tau} \{\Delta_0'(\tau)\} = \Delta r_{FW}(D) - \Delta r_{FW}(D - 1) \cong 10 \text{ bps}.$$

The resulting shortfalls are shown in Fig. 4 for the 13 firms in the sample¹¹.

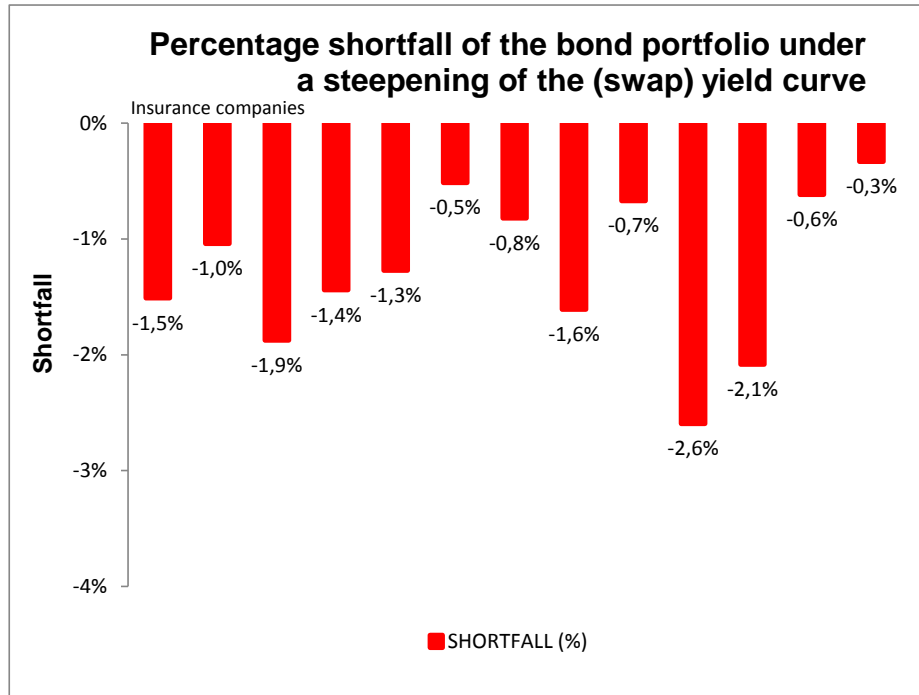
They highlight that, under the assumed scenario of instantaneous shock (i.e. non parallel, steepening shift), the insurance firms would stay relatively stable given that their bond portfolios would experience, at the liability horizon, downward variations within a range between -0.3% and -2.6%, these shortfalls being largely independent of the degree of duration mismatch. Consequently, the duration matching is not a guarantee of immunization.

FIG. 3 Duration matching in the considered sample of insurance companies



¹¹ The results hold true as an approximation because of the considered restrictive assumptions (single liability, partial portfolio, approximated duration matching etc.). Details on our input data and calculations are provided in Appendix B.

FIG. 4 Lower bound capital loss from the steepening of the term structure



4 Active strategy: the risk-return tradeoff optimization

Following Fong and Vasicek (1983 b), a more general approach in terms of risk-return optimization problem (analogous to Markowitz's mean-variance analysis) could be set in which the objective function is a general utility function of the expected return and risk of the bond portfolio.

Let $\bar{R}_0(H)$ be the current annual ex-ante return over the horizon H. By definition:

$$(1 + \bar{R}_0(H))^H \equiv \frac{A_0(H)}{A(0)}$$

In case of no shift of the term structure, the ex-ante is also the ex-post rate of return.

As shown in the Appendix A, after an instantaneous non constant shift of the term structure we have:

$$\Delta R_0(H) \simeq \frac{1}{H} \frac{\Delta A_0(H)}{A_0(H)} \simeq \frac{1}{H} M_0^2(H) \Delta S_0(H)$$

where:

$$\Delta S_0(H) \equiv \frac{1}{2} [\Delta_0^2(H) - \Delta_0'(H)] \geq 0$$

is a special function of the term structure shift from time 0 to time 0^+ and maturity H (see Appendix B).

Note that this function is the sum of a “shift in level” component (“convexity effect”), always positive, Δ^2 , and a “slope of shift” component Δ' , (“risk effect”) of ambiguous sign. In case of adverse shift ($\Delta_0^2 < \Delta_0'$, $\Delta S_0 < 0$) the realized return will be under the target value.

The portfolio risk can be measured by the volatility (standard deviation) of $\Delta \bar{R}_0(H)$ and it is proportional to the dispersion measure M_0^2 . As explained by Fong and Vasicek (1983 b, p. 235): “A portfolio with half the value of M^2 than another portfolio can be expected to produce half the dispersion of realized returns around the target value when submitted to a variety of interest rate scenarios than the other portfolio”.

Summing up (or integrating) all the shape changes between 0 and H-1 we have:

$$R_H(H) - \bar{R}_0(H) = \sum_{t=0}^{H-1} \Delta R_t(H) \simeq \frac{1}{H} \sum_{t=0}^{H-1} M_t^2(H) \Delta S_t(H)$$

where the difference, $R_H - \bar{R}_0$, between the future annual rate $\left(\frac{A_H(H)}{A(0)}\right)^{\frac{1}{H}} - 1$ and current forward rate is a random variable (sum of H random variables $\Delta S_t(H)$), with mean and variance $\mu_H \equiv E(R_H - \bar{R}_0)$, $\sigma_H^2 \equiv Var(R_H - \bar{R}_0)$.

In practice, in all financial markets, cash flows can be typically bought and sold only in pre-defined “bundles” (the coupon bonds) so that optimal management must be set in terms of available bond portfolios.

Given a set of K different bonds, the portfolio return can be represented as a linear combination of K random variables:

$$\sum_{i=1}^K w_i (R_{iH} - \bar{R}_{i0})$$

with mean μ_{iH} and covariance $\sigma_{ij} = Cov(R_{iH} - \bar{R}_{i0}, R_{jH} - \bar{R}_{j0})$

Clearly:

$$R_{iH} - \bar{R}_{i0} = \frac{1}{H} \sum_{t=0}^{H-1} M_{it}^2(H) \Delta S_t(H) \quad i=1, \dots, K$$

and we link the dynamics of M_{it}^2 and ΔS_t to their current values M_{i0}^2 and ΔS_0 as follows (see Fong and Vasicek, 1983 b):

$$M_{it}^2(H) \Delta S_t(H) \simeq M_{i0}^2(H) \left(\frac{H-t}{H}\right)^3 \Delta S_0(D_{i0}) \left(\frac{H-t}{D_{i0}}\right)$$

where, if D_{0i} is the i -th bond duration, we have¹²:

$$M_{i0}^2(H) = \sum_{j=1}^m (s_j - H)^2 w_{0j} = M_0^2(D_{i0}) + (D_{i0} - H)^2$$

so that¹³:

$$\mu_{iH} = E(R_{iH} - \bar{R}_{i0}) = \frac{1}{H} \sum_{t=0}^{H-1} M_{it}^2(H) E(\Delta S_t(H)) = \frac{1}{H} M_{i0}^2(H) E(\Delta S_0(D_{i0})) \frac{1}{D_{i0}} \sum_{t=0}^{H-1} \frac{(H-t)^4}{H^3},$$

$$\begin{aligned} \sigma_{ij} &= Cov(R_{iH} - \bar{R}_{i0}, R_{jH} - \bar{R}_{j0}) = Cov\left(\frac{1}{H} \sum_{t=0}^{H-1} M_{it}^2(H) \Delta S_t(H), \frac{1}{H} \sum_{v=0}^{H-1} M_{jv}^2(H) \Delta S_v(H)\right) \\ &= \frac{1}{H^2} M_{i0}^2(H) M_{j0}^2(H) \left(\sum_{t=0}^{H-1} \frac{(H-t)^4}{H^3}\right)^2 \frac{1}{D_{i0}} \frac{1}{D_{j0}} Cov(\Delta S_0(D_{i0}), \Delta S_0(D_{j0})) \end{aligned}$$

The active strategy is to find the portfolio w_i maximizing the objective function U , in mean and variance:

$$\begin{cases} \max_{w_i} U(\mathbf{w}'\boldsymbol{\mu}, \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}) \\ \sum_{1 \leq i \leq K} w_i \cdot D_{i0} = H \text{ (target duration constraint)} \\ \sum_{1 \leq i \leq K} w_i = 1 \text{ (budget constraint)} \\ w_i \geq 0 \text{ (no short selling)} \end{cases}$$

where $\boldsymbol{\mu}$ is the vector of expected returns μ_{iH} and $\boldsymbol{\Sigma} \equiv [\sigma_{ij}]$ is the covariance matrix.

As an example:

$$U(\mathbf{w}'\boldsymbol{\mu}, \sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}; \varphi) = \mathbf{w}'\boldsymbol{\mu} - \varphi \sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}$$

where the confidence level parameter φ (implying the maximization of the *lower bound* of the return confidence interval) has the same role of the risk aversion parameter in Markowitz

¹² Note that using (realistically) bonds instead of cash flows has no effect on the calculation of duration but it affects the calculation of risk. In fact, the duration of a portfolio of bonds is simply the “portfolio” (average) of bond durations; however, the M^2 of a portfolio of bonds is the “portfolio” of M^2 (within variance) plus the variance of durations (between variance). See the Appendix B.

¹³ Note that Fong and Vasicek (1983 b) assume, in the variance calculation, of the asymptotic approximation:

$$\sum_{t=0}^H M_t^4 = \frac{M_0^4}{H^6} \sum_{t=0}^H (H-t)^6 = \frac{M_0^4}{H^6} \frac{H(H+1)(2H+1)(3H^4+6H^3-3H+1)}{42} \xrightarrow{\text{for } H \rightarrow \infty} \frac{H}{7} M_0^4$$

so that the variance is proportional to $\frac{M_0^4}{7H}$. We do not use this asymptotic approximation.

risk-return approach: the higher the parameter, the more relevant the effect of portfolio variance and the more conservative the preferred portfolio will be. For $\varphi = 0$ the optimal portfolio will maximize the ex ante rate of return; for φ large, the optimal portfolio is the passive portfolio with minimum immunizing risk.

As in the classical Markowitz approach, the solution procedure can be divided in steps, the first one being the minimum variance portfolio for a given ex ante differential return m and the given horizon H :

$$\begin{cases} \min_{w_i} \sqrt{\mathbf{w}' \boldsymbol{\Sigma} \mathbf{w}} \\ \sum_{1 \leq i \leq K} w_i \cdot \mu_{iH} = m \\ \sum_{1 \leq i \leq K} w_i \cdot D_{i0} = H \text{ (target duration constraint)} \\ \sum_{1 \leq i \leq K} w_i = 1 \text{ (budget constraint)} \\ w_i \geq 0 \text{ (no short selling)} \end{cases}$$

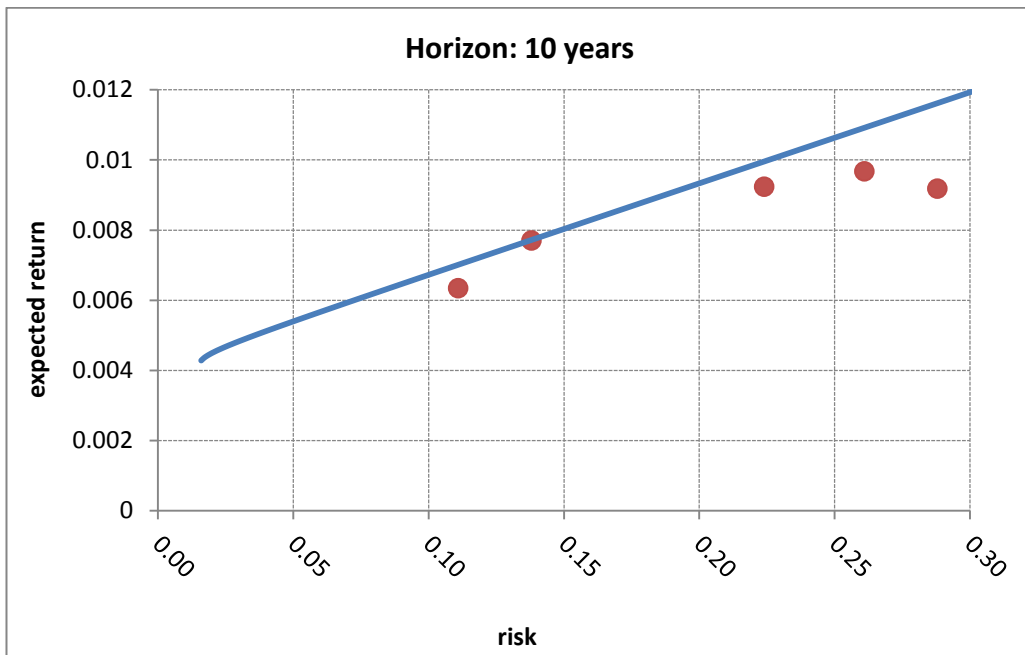
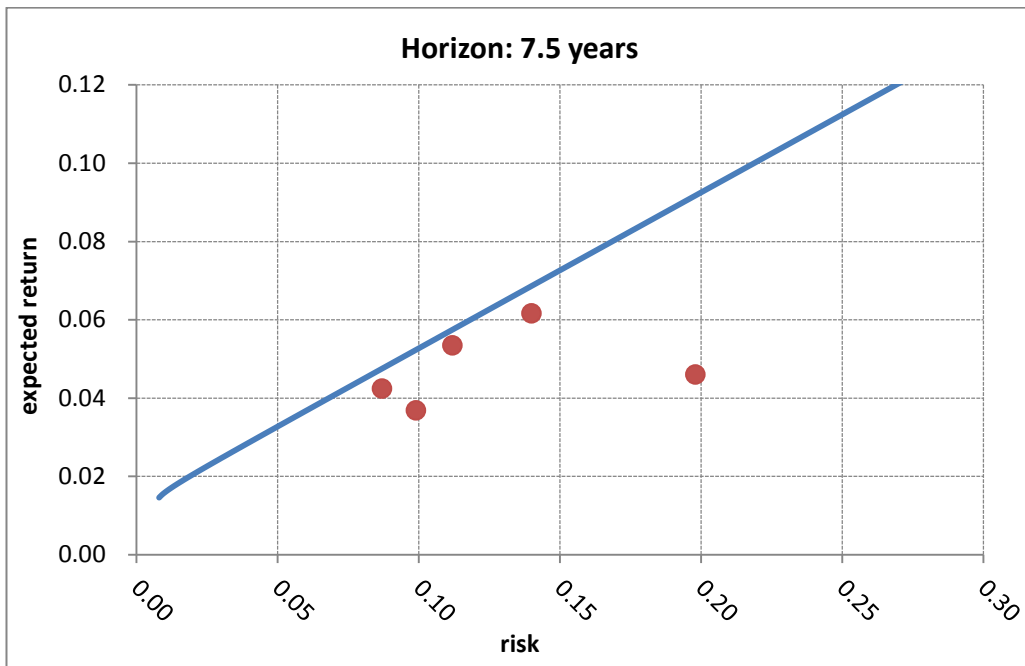
By varying m , the efficient frontier $m(\sigma)$ can be traced, for a given H .

As an example, the following frontiers (Fig. 5) have been obtained by simplifying the Government bond market into four bond benchmarks with maturity 3, 5, 10 and 20 years respectively (see Appendix C) and by calculating the efficient frontiers with horizon (average duration) $H=7.5$ (see Fig. 1) and $H=10$.

Ten companies in our sample have approximately the given average durations and they lie somehow below the frontier. Considering that the model, by construction, cannot measure the “selection” effect of asset management, the vertical gap between each firm and the frontier is a measure of the “allocation” inefficiency of the firm’s portfolio.

In general, assuming the horizon H as an additional control variable, an efficient surface $m(\sigma, H)$ could be obtained, showing the portfolio possibilities for an “active” management strategy in terms of optimal return-risk-horizon trade-off.

FIG. 5 Efficient frontiers for immunized portfolios at different horizons



5 Conclusions and further developments

Following the seminal work by Fong and Vasicek (1983, 1984), it is possible to actively manage the “immunization risk” of a duration-matching portfolio, i.e. a portfolio which, notwithstanding the equality of asset and liability durations (zero-duration gap or immunized portfolio), it has a risky expected return, being exposed to the effects of arbitrary shifts of the term structure of interest rates.

This risk is proportional to a measure (M^2) of the dispersion of the cash flow dates around the duration average so that, in this risk-return framework, you can build a generalized optimal management of the company bond portfolio essentially in the same way as asset managers choose the optimal portfolio along Markowitz’s efficient frontier.

The empirical application of the model shows that it can disclose how distant the actual portfolio is from the efficient frontier, helping to select, along the frontier, the optimal bond portfolio corresponding to the firm’s risk appetite.

Extensions of the analysis are manifold. Theoretically, one can attempt to consider the multiple-liability framework in order to be more consistent in the modeling of the life insurance business. Moreover, the shocks to the forward rates can be assumed to have an explicit stochastic dynamics, to be exploited in the calculation of moments. At the same time, the empirical implementation could take into account the complete set of bonds available in the financial market, in order to have more refined measures of risks, returns and the frontier.

The calculation of a three-dimensional frontier (risk, return and horizon) could be easily implemented through this more refined modelling and it could be proved useful in generalizing the traditional profit testing activity for new insurance products.

Appendix A – The classical theory of immunization

Let A_j for $j=1, \dots, m$ be the cash flows of a bond portfolio available at time s_j , $j=1, \dots, m$, $0 < s_1 < \dots < s_m$. Let $P(0, s)$ be the initial (time 0) discount function for maturity s .

The current value of the assets is:

$$A(0) = \sum_{j=1}^m A_j P(0, s_j)$$

where

$$P(0, s) = \exp\left(-\int_0^s r_{FW}(0, \tau) d\tau\right) = \exp(-sR(0, s))$$

$r_{FW}(0, \tau)$ initial instantaneous forward rate:

$$r_{FW}(0, \tau) \equiv -\frac{\partial \ln P(0, \tau)}{\partial \tau}$$

and $R(0, s)$ is the initial spot rate for maturity s :

$$R(0, s) = \frac{1}{s} \int_0^s r_{FW}(0, \tau) d\tau$$

Let \bar{L} be the value of a single liability at the target time H , such that¹⁴:

$$A(0) = L(0) = \bar{L} \exp\left(-\int_0^H r_{FW}(0, \tau) d\tau\right) \quad (\text{budget constraint})$$

Note that $\bar{L} \equiv A_0(H)$ is the forward, time H value of the portfolio calculated at time 0:

$$\bar{L} \equiv A_0(H) = A(0) \exp\left(\int_0^H r_{FW}(0, \tau) d\tau\right) = \sum_{j=1}^m A_j \exp\left(\int_{s_j}^H r_{FW}(0, \tau) d\tau\right)$$

If forward interest rates do not change, the value of the bond portfolio at the target date is $A(H) = \bar{L}$. If, instead, interest rates do change we could have $A(H) \geq \bar{L}$.

In fact:

¹⁴ This multiple asset – single liability case is equivalent to the case of an asset only bond portfolio with target horizon H and target value $\bar{L} = A(0) \exp\left(\int_0^H r_{FW}(0, \tau) d\tau\right)$.

$$A(H) = \sum_{j=1}^m A_j P(H, s_j) = \sum_{j=1}^m A_j \exp\left(\int_{s_j}^H r_{FW}(H, \tau) d\tau\right) = A_H(H)$$

The balance sheet is said to be **immunized against interest rate movements** if $A(H) \geq \bar{L}$ and it would be interesting to find the conditions (if any) under which a given balance sheet has this property.

Clearly, the immunizing conditions depend on the assets and liabilities as well as the type of movements assumed for the interest rates.

Theorem 1 (constant shift)

If the term structure of interest rates undergoes an additive and small constant shift from $r_{FW}(0, \tau)$ to $r_{FW}(0^+, \tau) = r_{FW}(0, \tau) + \Delta_0$ then the balance sheet with liability \bar{L} at time H (or the portfolio at the target horizon H) is immunized if the duration $D(A)$ of the portfolio is equal to H .

Proof. From the current budget constraint:

$$\sum_{j=1}^m A_j \exp\left(-\int_0^{s_j} r_{FW}(0, \tau) d\tau\right) = \bar{L} \exp\left(-\int_0^H r_{FW}(0, \tau) d\tau\right)$$

or equivalently

$$\sum_{j=1}^m A_j \exp\left(\int_{s_j}^H r_{FW}(0, \tau) d\tau\right) \equiv \sum_{j=1}^m a_j = \bar{L}$$

Let

$$G(r_{FW} + \Delta_0) \equiv \sum_{j=1}^m A_j \exp\left(\int_{s_j}^H (r_{FW}(0, \tau) + \Delta_0) d\tau\right) - \bar{L} \equiv \Delta A_0(H)$$

We have:

$$\frac{\partial G}{\partial \Delta_0} = \sum_{j=1}^m A_j (H - s_j) \exp\left(\int_{s_j}^H (r_{FW}(0, \tau) + \Delta_0) d\tau\right)$$

$$\frac{\partial^2 G}{\partial \Delta_0^2} = \sum_{j=1}^m A_j (H - s_j)^2 \exp\left(\int_{s_j}^H (r_{FW}(0, \tau) + \Delta_0) d\tau\right) > 0$$

so that $G(r_{FW} + \Delta_0) = 0$ for $\Delta_0 = 0$ and $G(r_{FW} + \Delta_0) \geq 0$ for every Δ_0 in a neighbourhood of 0, if $\Delta_0 = 0$ is a (local) minimum i.e. if:

$$\frac{\partial G}{\partial \Delta_0} \Big|_{\Delta_0=0} = \sum_{j=1}^m A_j (H - s_j) \exp\left(\int_{s_j}^H r_{FW}(0, \tau) d\tau\right) = 0$$

or

$$\frac{\sum_{j=1}^m A_j s_j \exp\left(\int_{s_j}^H r_{FW}(0, \tau) d\tau\right)}{\sum_{j=1}^m A_j \exp\left(\int_{s_j}^H r_{FW}(0, \tau) d\tau\right)} \equiv \frac{\sum_{j=1}^m a_j s_j}{\sum_{j=1}^m a_j} = H$$

Multiplying and dividing by $P(0, H)$ we have the duration condition¹⁵:

$$D(A) = \frac{\sum_{j=1}^m A_j s_j \exp\left(-\int_0^{s_j} r_{FW}(0, \tau) d\tau\right)}{\sum_{j=1}^m A_j \exp\left(-\int_0^{s_j} r_{FW}(0, \tau) d\tau\right)} = H \quad \blacksquare$$

This result is known as Fisher and Weil (1971) theorem.

In simple words, the duration condition guarantees that, after the assumed constant shift, the gain (loss) obtained through the reinvestment effect of short term cash flows is not smaller (not greater) than the loss (gain) obtained via the price effect of the long term asset component of the portfolio.

The special case of a flat term structure applied to the balance sheet of the life business of an insurance company (with multiple liability cash flows) had already been obtained by Redington (1952), who introduced also the term immunization “to signify the investment of the assets in such a way that the existing business is immune to a general change in the rate of interest” (ivi, p. 289).

Samuelson (1945) provided a similar proof with application to the US banking system.

The Redington’s “mean term” and the Samuelson’s “average time period” can be acknowledged as the Macaulay (1938) duration¹⁶.

Note that the duration condition must be maintained over time, so that immunization is in fact a dynamic process.

In terms of rate of returns, $(1 + \bar{R}_0)^H \equiv \frac{\bar{L}}{L(0)} \equiv \frac{A_0(H)}{A(0)}$ so that \bar{R}_0 is the ex-ante target return

evaluated at time 0, also obtained ex-post in case of stable interest rates. Taking logs and differentiating:

$$\Delta \bar{R}_0 \simeq \frac{1}{H} \frac{\Delta A_0(H)}{A_0(H)}$$

¹⁵ Note that the second order (convexity) condition is always satisfied in the single liability case.

¹⁶ The duration measure was independently obtained by Hicks (1939) with the name of “average period”. Earlier developments could be seen in Lidstone (1893).

and, under the Fisher-Weil hypotheses, $\Delta \bar{R}_0 \geq 0$ for any small, additive shift of the term structure.

Fisher-Weil theorem has been generalized in terms of non-constant shift (Fong and Vasicek, 1984, Shiu, 1987, 1990, Montrucchio and Peccati, 1991).

Theorem 2 (non constant shift)

If the term structure of interest rates undergoes an additive and non constant shift from $r_{FW}(0, \tau)$ to $r_{FW}(0^+, \tau) = r_{FW}(0, \tau) + \Delta_0(\tau)$ then the balance sheet with liability \bar{L} at time H (or the portfolio at the target horizon H) is immunized if the duration $D(A)$ of the portfolio is equal to H (duration matching condition) and $\Delta_0^2(\tau) \geq \Delta_0'(\tau)$ for every τ (“convexity condition”).

Proof. Define:

$$f(s) \equiv \exp\left(\int_s^H \Delta_0(\tau) d\tau\right)$$

so that:

$$f(H) = 1, \quad f'(s) = -f(s)\Delta_0(s), \quad f''(s) = f(s)[\Delta_0^2(s) - \Delta_0'(s)] \geq 0 \text{ (by the “convexity condition”)}$$

Then the change in value is:

$$\Delta A_0(H) \equiv G(r_{FW} + \Delta_0) - G(r_{FW}) = G(r_{FW} + \Delta_0) = \sum_{j=1}^m a_j (f(s_j) - 1)$$

and by Taylor (exact) formula:

$$f(s_j) = f(H) + (s_j - H)f'(H) + \frac{1}{2}(s_j - H)^2 f''(\xi_j)$$

so that, substituting:

$$\begin{aligned} G(r_{FW} + \Delta_0) &= -\sum_{j=1}^m a_j (s_j - H)\Delta_0(H) + \frac{1}{2} \sum_{j=1}^m a_j (s_j - H)^2 f(\xi_j) [\Delta_0^2(\xi_j) - \Delta_0'(\xi_j)] \\ &= \frac{1}{2} \sum_{j=1}^m a_j (s_j - D)^2 f(\xi_j) [\Delta_0^2(\xi_j) - \Delta_0'(\xi_j)] \geq 0 \end{aligned} \tag{A.1}$$

where the second equality comes from the duration matching hypothesis $D=H$ and the inequality from the “convexity condition”. ■

In general, if the term structure shift is not convex, the change in the forward value will be negative.

As an alternative proof (Montrucchio and Peccati, 1991), define $g \equiv f-1$, so that if f is convex, g is convex. Then:

$$\frac{G(r_{FW} + \Delta_0)}{\bar{L}} = E(g(\tilde{X}))$$

where \tilde{X} is the random variable taking value s_j with probability $\frac{a_j}{\bar{L}}$.

By Jensen inequality:

$$\frac{G(r_{FW} + \Delta_0)}{\bar{L}} = E(g(\tilde{X})) \geq g(E(\tilde{X})) = g(H) = 0 \quad \blacksquare$$

Note that the $\Delta_0(\tau)$ is the term structure of the change in the forward rates $r_{FW}(0, \tau)$ so that $\Delta_0'(\tau)$ is the slope of this change and the convexity conditions is satisfied if $\Delta_0(\tau)$ is monotone decreasing (smaller changes the longer the maturity). More complex cases are given in Appendix B.

An approximation is obtained using the Taylor formula around H:

$$\begin{aligned} f(s_j) &\approx f(H) + (s_j - H)f'(H) + \frac{1}{2}(s_j - H)^2 f''(H) \\ &= 1 - (s_j - H)\Delta_0(H) + \frac{1}{2}(s_j - H)^2 [\Delta_0^2(H) - \Delta_0'(H)] \end{aligned}$$

so that, for H=D:

$$\frac{\Delta A_0(H)}{A_0(H)} \equiv \frac{G(r_{FW} + \Delta_0)}{\bar{L}} \approx \frac{1}{2\bar{L}} \sum_{j=1}^m a_j (s_j - D)^2 [\Delta_0^2(D) - \Delta_0'(D)] \equiv \frac{1}{2} M_0^2(D) [\Delta_0^2(D) - \Delta_0'(D)]$$

where $M_0^2(D)$ is a variability measure of portfolio cash-flow times measured at time 0:

$$M_0^2(D) \equiv \frac{\sum_{j=1}^m a_j (s_j - D)^2}{\sum_{j=1}^m a_j} = \frac{\sum_{j=1}^m A_j (s_j - D)^2 \exp\left(-\int_0^{s_j} r_{FW}(0, \tau) d\tau\right)}{\sum_{j=1}^m A_j \exp\left(-\int_0^{s_j} r_{FW}(0, \tau) d\tau\right)}$$

Note that M^2 is a proper variance of times, given that D is the time average (duration). This implies $M^2 = C - D^2$ where C is the asset convexity (average of times squared).

The total effect is the sum of a “shift in level” component or convexity effect, always positive, Δ^2 , and a “slope of shift” component Δ' , or risk effect, of ambiguous sign. Under the traditional approach of a constant, parallel shift of the term structure ($\Delta = \text{const}$, $\Delta' = 0$) the total effect is

due to convexity only and it is always positive, so that the burbell portfolio would gain an extra-return¹⁷.

A lower bound for the change in value was obtained by Fong and Vasicek (1984)

Theorem 3 (lower bound for non constant shift)

If the term structure of interest rates undergoes an additive and non constant shift from $r_{FW}(0,\tau)$ to $r_{FW}(0^+,\tau) = r_{FW}(0,\tau) + \Delta_0(\tau)$ and if the duration $D(A)$ of the portfolio is equal to the maturity H of the liability \bar{L} (duration matching condition), then a lower bound for the change in value of the portfolio is:

$$\frac{\Delta A_0(H)}{A_0(H)} \equiv \frac{G(r_{FW} + \Delta_0)}{\bar{L}} \geq -\frac{1}{2} K_0 M_0^2(D)$$

where K_0 is the maximum of $\Delta_0'(\tau)$ and $M_0^2(D)$ was defined above.

Proof. From the first order approximation for $f(s) = e^{x(s)} \geq 1 + x(s)$ we have:

$$G(r_{FW} + \Delta_0) = \sum_{j=1}^m a_j (f(s_j) - 1) \geq \sum_{j=1}^m a_j \int_{s_j}^H \Delta_0(\tau) d\tau$$

By the integral formula:

$$\Delta_0(H) - \Delta_0(\tau) = \int_{\tau}^H \Delta_0'(u) du$$

so that, integrating and using the Fubini theorem to change the order of integration:

$$\begin{aligned} \int_{s_j}^H \Delta_0(\tau) d\tau &= (H - s_j) \Delta_0(H) - \int_{s_j}^H \int_{\tau}^H \Delta_0'(u) du d\tau = (H - s_j) \Delta_0(H) - \int_{s_j}^H \Delta_0'(u) \int_{s_j}^u d\tau du \\ &= (H - s_j) \Delta_0(H) - \int_{s_j}^H \Delta_0'(u) (u - s_j) du \end{aligned}$$

If $K_0 \equiv \max_u \{\Delta_0'(u)\}$

$$\int_{s_j}^H \Delta_0(\tau) d\tau \geq (H - s_j) \Delta_0(H) - K_0 \int_{s_j}^H (u - s_j) du = (H - s_j) \Delta_0(H) - \frac{1}{2} K_0 (H - s_j)^2$$

so that, using the duration matching condition $D=H$:

$$G(r_{FW} + \Delta_0) \geq -\frac{1}{2} K_0 \sum_{j=1}^m a_j (D - s_j)^2 = -\frac{1}{2} K_0 M_0^2(D) \bar{L} \quad \blacksquare$$

For this proof, Shiu (1987) uses the exact Taylor formula:

¹⁷ Clearly, interest rate dynamics implying constant shifts are not arbitrage-free.

$$v(s_j) \equiv \int_{s_j}^H \Delta_0(\tau) d\tau = v(H) + (s_j - H)v'(H) + \frac{1}{2}(s_j - H)^2 v''(\xi_{s_j})$$

$$v'(s_j) = -\Delta_0(s_j)$$

$$v''(s_j) = -\Delta_0'(s_j)$$

Nawalkha and Chambers (1997) use higher order approximations for $f(s)$ obtaining:

$$G(r_{FW} + \Delta_0) \cong -\bar{L} \sum_{q=1}^Q \frac{1}{q!} M_0^q(D) X_q(D)$$

where

$$M_0^q(D) \equiv \frac{\sum_{j=1}^m a_j (s_j - D)^q}{\sum_{j=1}^m a_j} = \frac{\sum_{j=1}^m A_j (s_j - D)^q \exp\left(-\int_0^{s_j} r_{FW}(0, \tau) d\tau\right)}{\sum_{j=1}^m A_j \exp\left(-\int_0^{s_j} r_{FW}(0, \tau) d\tau\right)}$$

and $X_q(\tau)$ is a polynomial function of the first $q-1$ derivatives of $\Delta_0(\tau)$.

The extension to multiple liabilities was firstly considered in the Redington seminal paper. He expressed financial immunization by introducing (sufficient) conditions for a fixed income portfolio backing insurance commitments (spread across many maturities) to be immunized (i.e., protected) against parallel and small variations of a flat term structure.

He proved that assets remain above liabilities in value if, subject to the initial balance constraint, the duration (time average) of asset cash flows equals the duration of liabilities and the convexity (time dispersion) of asset cash flows is greater than that of liability cash flows.

As duration is the first derivative with respect to the interest rate, convexity represent the second derivative.

This multiple liability immunization has been generalized by Shiu (1988).

Appendix B – The shift function

The Fong-Vasicek (1984) sufficient condition for immunization is a constraint on the shift function $\Delta_t(\tau) = r_{FW}(t+dt, \tau) - r_{FW}(t, \tau)$:

$$\Delta_t^2(\tau) - \Delta_t'(\tau) = c \geq 0 \quad \forall \tau$$

Note that the term “convexity condition” stems from the fact that under this condition the function:

$$f(s) \equiv \exp\left(\int_s^H \Delta_t(\tau) d\tau\right)$$

is convex ($f''(s) \geq 0$).

Omitting the subscript, we can check that $\Delta(\tau) = \text{const}$ (uniform additive positive or negative shift) satisfies the convexity condition.

The same holds for any shift for which $\Delta'(\tau)$ is negative (decreasing term structure of shifts): for example $\Delta(\tau) = \exp(-a\tau)$ for $a = \nu c \geq 0$.

In general, the convexity condition is an ordinary, non linear differential equation of the Riccati type.

Using the substitution $\Delta = -u'/u$ we obtain:

$$u''(\tau) - c(\tau)u(\tau) = 0$$

The special case $c(\tau) = \tau$ is well known because one solution is the Airy function (Abramowitz and Stegun (eds.), 1964, ch. 10, p.446):

$$u(\tau) = \frac{\sqrt{\tau}}{3} \left[I_{-\frac{1}{3}}\left(\frac{2}{3}\tau^{3/2}\right) - I_{\frac{1}{3}}\left(\frac{2}{3}\tau^{3/2}\right) \right]$$

where $I_\nu(x)$ is the modified Bessel function of order ν (Abramowitz and Stegun (eds.) 1964, ch. 9 p. 375).

The integral representation is:

$$u(\tau) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{x^3}{3} + x\tau\right) dx$$

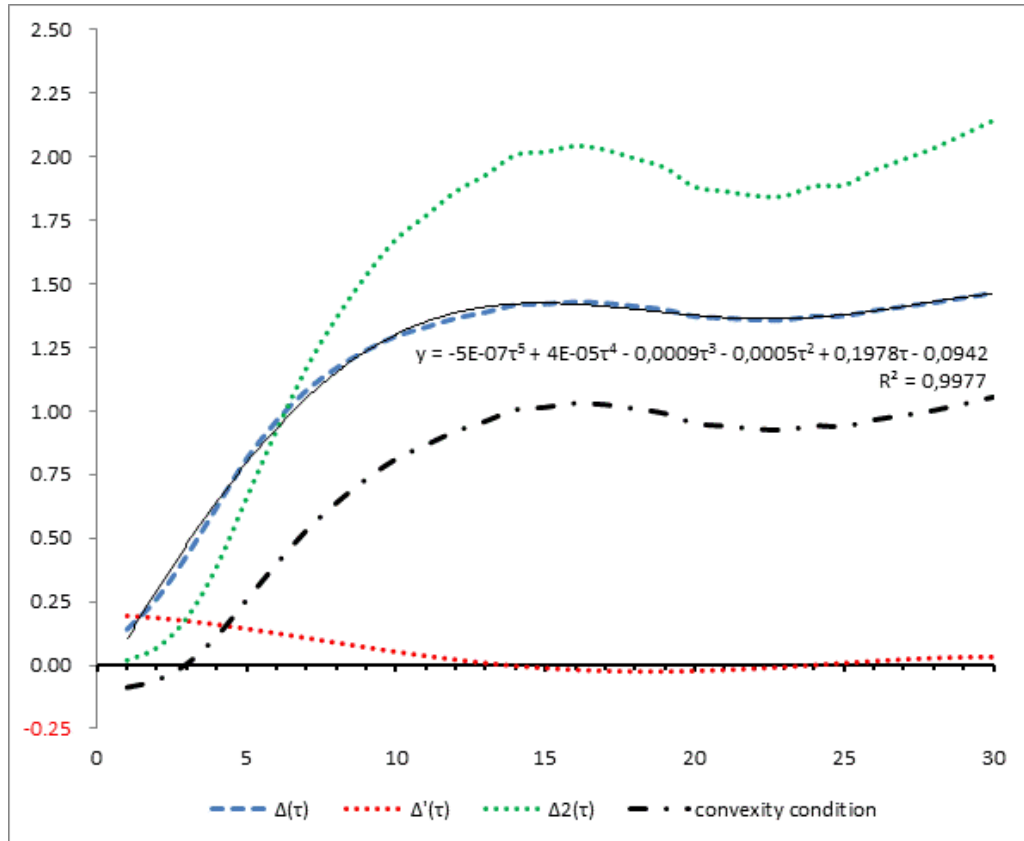
so that:

$$\Delta(\tau) = \frac{\int_0^\infty \sin\left(\frac{x^3}{3} + x\tau\right) x dx}{\int_0^\infty \cos\left(\frac{x^3}{3} + x\tau\right) dx}$$

In the stochastic case, the shift function is the stochastic change in the forward rate $dr_{FW}(t, \tau)$. In the Vasicek term structure model it is given by $e^{-k\tau} dr(t)$, i.e. the change in the short term rate smoothed by a negative exponential.

Using the shift function implied by the EIPOA Stress Test (see Fig. 2) we obtain the convexity condition displayed in Fig. A1.

FIG. A1 EIOPA shift function $\Delta(\tau)$ and convexity condition



Appendix C – The stochastic approach to immunization

Under the no-arbitrage approach of financial markets, the case of “flat shifts” and the immunization property $A(H) \geq \bar{L}$ imply a riskless arbitrage and are therefore not compatible with equilibrium (Boyle, 1978, Ingersoll, Skelton and Weil, 1978).

In continuous time, under stochastic term structure models, the immunization condition means that the value of assets in the next instant must be equal to the value of liabilities, so that, in differential terms: $dA(t) = dL(t)$ and immunization is guaranteed if the value of assets and liabilities have the same sensitivity to the state variables (Albrecht, 1985). This has suggested the definition of a generalized or stochastic duration concept (Cox, Ingersoll and Ross, 1979).

As an example of this stochastic approach to the asset-liability problem, let us consider the case of the extended-Vasicek, no arbitrage term structure model (Hull and White, 1990):

$$dr(s) = [\hat{a}(s) - kr(s)]ds + \sigma d\hat{Z}(s)$$

where the time-dependent drift $\hat{a}(s)$ is obtained in order to guarantee a perfect fit of the theoretical term structure with the current yield curve:

$$\hat{a}(s) = kr_{FW}(t, s) + \frac{\partial r_{FW}(t, s)}{\partial s} + \frac{\sigma^2}{2k}(1 - e^{-2k(s-t)})$$

The solution for $r(s)$ is:

$$r(s) = r_{FW}(t, s) + \frac{\sigma^2}{2}G^2(t, s) + \sigma \int_t^s e^{-k(s-u)} d\hat{Z}(u)$$

$$G(t, s) = \frac{1 - e^{-k(s-t)}}{k}$$

and the relative difference between ex post and ex ante portfolio value is:

$$\begin{aligned}
\frac{A(H) - A_0(H)}{A_0(H)} &= \frac{\sum_{j=1}^m A_j \exp\left(\int_{s_j}^H r(u) du\right) - \sum_{j=1}^m A_j \exp\left(\int_{s_j}^H r_{FW}(0, \tau) d\tau\right)}{\sum_{j=1}^m A_j \exp\left(\int_{s_j}^H r_{FW}(0, \tau) d\tau\right)} \\
&= \frac{\sum_{j=1}^m A_j \exp\left(\int_{s_j}^H r_{FW}(0, \tau) d\tau\right) \left[\exp\left(\int_{s_j}^H r(u) du - \int_{s_j}^H r_{FW}(0, \tau) d\tau\right) - 1 \right]}{\sum_{j=1}^m A_j \exp\left(\int_{s_j}^H r_{FW}(0, \tau) d\tau\right)} \\
&\approx \frac{\sum_{j=1}^m a_j \left[\int_{s_j}^H (r(u) - r_{FW}(0, u)) du \right]}{\sum_{j=1}^m a_j} \equiv \sum_{j=1}^m b_j X_{0j}
\end{aligned}$$

The variable X_{ij} has an explicit form in this case:

$$X_{tj} = \int_{s_j}^H (r(u) - r_{FW}(t, u)) du = \frac{\sigma^2}{2k^2} \int_{s_j}^H (1 - e^{-k(u-t)})^2 du + \sigma \int_{s_j}^H \int_t^u e^{-k(u-v)} d\hat{Z}(v) du$$

with mean:

$$\begin{aligned}
E(X_{tj}) &= \frac{\sigma^2}{2k^2} \int_{s_j}^H (1 - e^{-k(u-t)})^2 du \\
&= \frac{\sigma^2}{k^3} \left[\frac{k}{2} (H - s_j) + e^{-k(H-t)} - e^{-k(s_j-t)} - \frac{e^{-2k(H-t)}}{4} + \frac{e^{-2k(s_j-t)}}{4} \right]
\end{aligned}$$

and covariance

$$\begin{aligned}
Cov(X_{ti}, X_{tj}) &= \sigma^2 E \left[\int_{s_i}^H \int_t^u e^{-k(u-v)} d\hat{Z}(v) du \int_{s_j}^H \int_t^s e^{-k(s-v)} d\hat{Z}(v) ds \right] \\
&= \sigma^2 \int_{s_i}^H \int_{s_j}^H \int_t^{\min(u,s)} e^{-k(u-v)-k(s-v)} dv ds du
\end{aligned}$$

so that a risk-return approach can be analytically developed.

Appendix D – Some financial computations

In this Appendix we give some technical details of the computations involving financial variables.

▪ The computation of term structure changes

In the empirical application we use daily data and we calculate, from the 1 year-forward rates, $r_{FW}(t, \tau, \tau+1)$ for $\tau=1, \dots, 25$ the daily changes:

$$\Delta_t(\tau) \equiv r_{FW}(t+1, \tau, \tau+1) - r_{FW}(t, \tau, \tau+1)$$

$$\Delta_t'(\tau) \equiv \Delta_t(\tau+1) - \Delta_t(\tau)$$

$$\Delta S_t(\tau) \equiv \frac{1}{2} [\Delta_t^2(\tau) - \Delta_t'(\tau)]$$

▪ The analytic, closed form computation of the portfolio cash-flow dispersion

It is possible to compute in closed-form, i.e. via an analytic formula, the cash-flow dispersion of a securities portfolio. Portfolio cash-flow dispersion turns out to be a quadratic function of portfolio duration, of its yield-to-maturity and its convexity¹⁸:

$$M^2(D) = C^{Mod} \cdot (1 + y)^2 - D \cdot (D + 1),$$

$M^2(D)$ = variance of the asset cash-flow maturities around their *duration*; y = portfolio *yield-to-maturity*, D = portfolio *duration* (average time-to-maturity), C^{Mod} = portfolio modified convexity.

The variance with respect to Horizon H is obtained as:

$$M^2(H) = M^2(D) + (D - H)^2$$

▪ Yield, duration and convexity aggregation at portfolio level

▪ The computation of the portfolio aggregate yield-to-maturity

Traditionally one computes the average portfolio yield, weighted by the amount invested (w.r.t. either book-value or market value). This estimate represents an approximation to the yield-to-maturity as seen from the valuation date. A better alternative seems to consist of computing an estimate that takes into account the different time distance of cash-flows from the valuation date: one weights each yield by the “dollar duration” of its security. One obtains the so called WADD yield (*Weighted Average Dollar Duration Yield*, see Grondin (1998)). The Dollar Duration is the product of *duration* and price of a security.

▪ The computation of average portfolio *Duration* and *Convexity*

The (modified) duration was downloaded from Bloomberg, for each security in the data set (at the date 31/12/2014). Portfolio duration is the average (weighted by market value amounts) of the durations of single securities.

¹⁸ See for example, Smith (2010), p. 1670, formula (9) or de La Grandville (2001), p. 163, formula (19). With continuous compounding, the formula simplifies into $M^2=C \cdot D^2$.

Convexity was also downloaded from Bloomberg, for each security (at the date 31/12/2014). Bloomberg approximated the second derivative (of price w.r.t. yield) as the “central difference” by varying yield-to-maturity by +/- 1 basis point, that is¹⁹:

$$C^{Mod} \cong \frac{1}{P} \cdot \frac{\frac{P^+ - P}{\Delta y} - \frac{P - P^-}{\Delta y}}{\Delta y} = \frac{1}{P} \cdot \frac{P^+ - 2 \cdot P + P^-}{(\Delta y)^2} = 10.000 \cdot \frac{P^+ - 2 \cdot P + P^-}{P}$$

$\Delta y = 0,01\%$

Convexity is a “quadratic” measure and hence it can be linearly aggregated (by summation): by weighting with the amounts (at market value) one obtains portfolio convexity.

- **The bond benchmarks for maturity classes**

The four bond classes used in the empirical application are represented by the following benchmark bonds:

Bond index	Bond ISIN	Time-to-maturity (years)	Duration (years)	Convexity	M-squared
3	IT0004867070	2.84	2.72	10.31	0.21
5	IT0003644769	5.09	4.58	27.08	1.54
10	IT0004513641	10.17	8.27	86.32	9.59
20	IT0003535157	19.60	13.46	241.60	46.84

- **M-squared for a portfolio of bonds**

Let $B_T = B_1 + B_2$ a portfolio of 2 bonds with cash flows c_{11} , c_{12} and c_{21} , c_{22} respectively at times t_{11} , t_{12} and t_{21} , t_{22} respectively.

Using a flat term structure:

$$B_1 = c_{11} \exp(-rt_{11}) + c_{12} \exp(-rt_{12})$$

$$B_2 = c_{21} \exp(-rt_{21}) + c_{22} \exp(-rt_{22})$$

$$D_1 = t_{11} c_{11} \exp(-rt_{11}) / B_1 + t_{12} c_{12} \exp(-rt_{12}) / B_1$$

$$D_2 = t_{21} c_{21} \exp(-rt_{21}) / B_2 + t_{22} c_{22} \exp(-rt_{22}) / B_2$$

$$\begin{aligned} D_T &= t_{11} c_{11} \exp(-rt_{11}) / B_1 B_1 / B_T + t_{12} c_{12} \exp(-rt_{12}) / B_1 B_1 / B_T \\ &\quad + t_{21} c_{21} \exp(-rt_{21}) / B_2 B_2 / B_T + t_{22} c_{22} \exp(-rt_{22}) / B_2 B_2 / B_T \\ &= D_1 B_1 / B_T + D_2 B_2 / B_T \end{aligned}$$

$$M_1^2 = (t_{11} - D_1)^2 c_{11} \exp(-rt_{11}) / B_1 + (t_{12} - D_1)^2 c_{12} \exp(-rt_{12}) / B_1$$

$$M_2^2 = (t_{21} - D_2)^2 c_{21} \exp(-rt_{21}) / B_2 + (t_{22} - D_2)^2 c_{22} \exp(-rt_{22}) / B_2$$

$$\begin{aligned} M_T^2 &= (t_{11} - D_T)^2 c_{11} \exp(-rt_{11}) / B_1 B_1 / B_T + (t_{12} - D_T)^2 c_{12} \exp(-rt_{12}) / B_1 B_1 / B_T \\ &\quad + (t_{21} - D_T)^2 c_{21} \exp(-rt_{21}) / B_2 B_2 / B_T + (t_{22} - D_T)^2 c_{22} \exp(-rt_{22}) / B_2 B_2 / B_T \\ &= [M_1^2 + (D_1 - D_T)^2] B_1 / B_T + [M_2^2 + (D_2 - D_T)^2] B_2 / B_T \end{aligned}$$

¹⁹ Bloomberg data include in the price P accrued interest (full price = clean price + accrued interest) and assume a variation of +/- 1 basis point w.r.t. the yield implicit in the price P .

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