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An introduction to Poisson processes and their generalizations

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An Introduction to Poisson processes and their generalisations

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Chapter 1

Poisson processes

1.1 The homogeneous Poisson process

An homogeneous Poisson process $N(t) = N((0,t]) = N(0,t]$, $t > 0$, is a continuous-time stochastic process taking integer values representing the number of events happening over a *finite* time interval $(0,t]$. Its probabilistic evolution is governed by the following rules:

Definition 1.

1. $\Pr\{N(0) = 0\} = 1$;
2. $\Pr\{N(t, t+dt] = 1\} = \lambda dt + o(dt)$, con $\lambda > 0$;
3. $\Pr\{N(t, t+dt] = 0\} = 1 - \lambda dt + o(dt)$;
4. $\Pr\{N(t, t+dt] > 1\} = o(dt)$;
5. If $0 = t_0 < t_1 < \ldots < t_n < t$, the random variables $N(t_{j-1}, t_j] = N(t_j) - N(t_{j-1})$, $1 \leq j \leq n$, are independent, i.e., the process has independent increments.

where $N(s,t]$ is the number of events in $(s,t]$ with $s < t$. Homogeneity means that the "intensity" or "rate" λ (i.e. number of events per unit time) is constant. Note, in particular, that $N(t) - N(s)$ is independent of $N(s) - N(0) = N(s)$.
The Poisson process describes the realisation of a flow of rare events. It is applied to a wide range of phenomena such as the arrival of customers in a bank agency, floods, earthquakes, car clashes, the number of particles emitted by a radioactive source, and so on (cf. Fig. 1.1).
From the previous assumptions we can extract several probabilistic information.

Theorem 1.1: The state probabilities $p_k(t) \equiv \Pr\{N(t) = k\}$, $k \geq 0$, $t \geq 0$, satisfy the following difference-differential equations:

$$\frac{d}{dt} p_k(t) = -\lambda p_k(t) + \lambda p_{k-1}(t), \qquad k \geq 0, t > 0 \tag{1.1}$$

with initial conditions

$$p_k(0) = \begin{cases} 1 & k = 0, \\ 0 & k > 0, \end{cases} \tag{1.2}$$

Clearly $\forall t \quad p_{-1}(t) = 0$. In order to derive the equations (1.1)-(1.2), we write

$$\begin{aligned}
p_k(t+dt) =& \Pr\{N(t+dt] = k\} \\
&= \Pr\Big\{(N(t) = k, N(t,t+dt] = 0) \cup (N(t) = k-1, N(t,t+dt] = 1) \\
&\qquad \textstyle\bigcup_{j=2}^{k} (N(t) = k-j, N(t,t+dt] = j)\Big\} \\
&= \Pr\{N(t) = k, N(t,t+dt] = 0\} + \Pr\{N(t) = k-1, N(t,t+dt] = 1\} \\
&\qquad + \sum_{j=2}^{k} \Pr\{N(t) = k-j, N(t,t+dt] = j\} \\
&= p_k(t)(1-\lambda dt) + p_{k-1}(t)\lambda dt + o(dt)
\end{aligned} \tag{1.3}$$

From (1.3), equation (1.1) immediately emerges.
We used the additive property of probability and properties 2, 3, 4 above.

Theorem 1.2: The distribution of the homogeneous Poisson process is given by

$$p_k(t) = \Pr\{N(t) = k \mid N(0) = 0\} = e^{-\lambda t}\frac{(\lambda t)^k}{k!}, \qquad k \geq 0,\ t > 0 \tag{1.4}$$

The simplest method to obtain (1.4) is the technique based on the probability generating function (for the p.g.f. see Appendix A)

$$G(u,t) = \sum_{k=0}^{\infty} u^k p_k(t), \qquad t{\geq}0,\ |u| \leq 1. \tag{1.5}$$

From (1.1) we can write

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{\partial p_k(t)}{\partial t} u^k &= \sum_{k=0}^{\infty} (-\lambda p_k(t) + \lambda p_{k-1}(t)) u^k \\
&= -\lambda G(u,t) + \lambda \sum_{k=0}^{\infty} p_{k-1}(t) u^k \\
&= -\lambda G(u,t) + \lambda \sum_{h=0}^{\infty} p_h(t) u^{h+1} \\
&= -\lambda G(u,t) + \lambda u G(u,t) = \lambda(u-1)G(u,t)
\end{aligned} \tag{1.6}$$

thus obtaining the Cauchy problem:

$$\begin{cases} \frac{\partial}{\partial t} G(u,t) = \lambda(u-1)G(u,t) \\ G(u,0) = 1. \end{cases} \tag{1.7}$$

The general solution of (1.7) is given by

$$G(u,t) = k e^{\lambda t(u-1)}, \tag{1.8}$$

and since

$$G(u,0) = 1 \tag{1.9}$$

we obtain that

$$G(u,t) = e^{\lambda t(u-1)} = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} u^k, \qquad t > 0, |u| \leq 1. \tag{1.10}$$

Comparing (1.5) and (1.10) we extract the distribution (1.4).

Remark 1.1. The Poisson distribution from the Binomial
It is possible to obtain the Poisson distribution as a "special case" of the Binomial distribution when $n \to \infty$, $p_n \to 0$ and $np_n = \lambda$
Indeed, being $p_n = \frac{\lambda}{n}$

$$\begin{aligned}\lim_{\substack{n\to\infty \\ p_n \to 0}} \tbinom{n}{k} {p_n}^k (1-p_n)^{n-k} &= \lim_{n\to\infty} \tfrac{n(n-1)\dots(n-k+1)}{k!} (\tfrac{\lambda}{n})^k (1-\tfrac{\lambda}{n})^{n-k} \\ &= \lim_{n\to\infty} \tfrac{n^k + O(n^{k-1})}{k!} (\tfrac{\lambda}{n})^k (1-\tfrac{\lambda}{n})^{n-k} \\ &= \lim_{n\to\infty} \tfrac{\lambda^k}{k!} \tfrac{(1-\frac{\lambda}{n})^n}{(1-\frac{\lambda}{n})^k} = \tfrac{\lambda^k}{k!} e^{-\lambda}\end{aligned}$$

Remark 1.2. Note that the *standard* Poisson process is a continuous-time, discrete-space process with independent increments such that
$N(0) = 0$ a.s.
$\Pr\{N(t) - N(s) = k\} \sim e^{-\lambda(t-s)} \frac{\lambda^k (t-s)^k}{k!} \qquad k = 0, 1, 2, \dots$

Theorem 1.3: An alternative method is to resolve the equations (1.1) recursively.
The solution of

$$\begin{cases} \frac{d}{dt} p_0(t) = -\lambda p_0(t), \\ p_0(0) = 1, \end{cases} \tag{1.11}$$

is $p_0(t) = e^{-\lambda t}$ with $t > 0$. If we suppose that

$$p_{k-1}(t) = e^{-\lambda t} \frac{(\lambda t)^{k-1}}{(k-1)!}, \quad k \geq 1 \tag{1.12}$$

we obtain the following non-homogeneous linear equation

$$\frac{d}{dt} p_k(t) = -\lambda p_k(t) + \lambda e^{-\lambda t} \frac{(\lambda t)^{k-1}}{(k-1)!} \tag{1.13}$$

with the initial condition $p_k(0) = 0$ for $k > 0$. The equation (1.13) has the form

$$y' + \alpha(x) y = \beta(x) \tag{1.14}$$

and its general solution reads

$$y(x) = e^{-\int_0^x \alpha(z)dz} \left\{ \int_0^x \beta(z) e^{\int_0^z \alpha(w)dw} + \text{const} \right\}. \tag{1.15}$$

So, after some calculation we have that

$$\begin{aligned} p_k(t) &= e^{-\lambda t}\left\{\lambda\int_0^t e^{\lambda s}e^{-\lambda s}\frac{(\lambda s)^{k-1}}{(k-1)!}ds + \text{const}\right\} \\ &= e^{-\lambda t}\left\{\frac{(\lambda t)^k}{k!} + \text{const}\right\} = e^{-\lambda t}\frac{(\lambda t)^k}{k!}, \end{aligned} \tag{1.16}$$

where the initial condition is $p_k(0) = 0$, $k \geq 1$, has been taken into account.

1.1.1 Properties of the homogeneous Poisson process

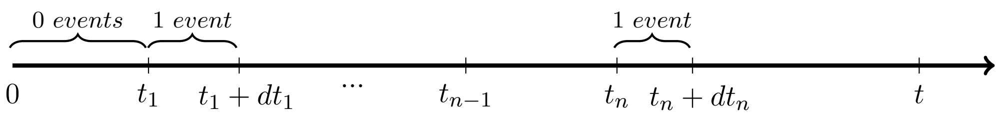
(i) Under the condition that $N(t) = n$, the distribution of the instants of occurrence of the n events $T_1, \cdots, T_n$ is

$$\Pr\{T_1 \in dt_1, \cdots, T_n \in dt_n \mid N(t) = n\} = \frac{n!}{t^n}dt_1 \cdots dt_n, \qquad \text{with } 0 < t_1 < t_2 < \cdots < t_n < t \tag{1.17}$$

Proof. From property 5 and formula (1.4) we have that

$$\begin{aligned} Pr\{T_1 \in dt_1, \cdots, T_n \in dt_n \mid N(t) = n\} &= \frac{Pr\{T_1 \in dt_1, \cdots, T_n \in dt_n, N(t) = n\}}{Pr\{N(t) = n\}} \\ = \frac{Pr\{N(0,t_1] = 0, N(t_1, t_1+dt_1] = 1, \cdots, N(t_{n-1}, t_n] = 0, N(t_n, t_n+dt_n] = 1, N(t_n+dt_n, t] = 0\}}{Pr\{N(t) = n\}} & \\ = \frac{\underbrace{e^{-\lambda t_1}}_{\text{no jumps}}\underbrace{\lambda dt_1}_{\text{jump}}\cdots\underbrace{e^{-\lambda(t_n-t_{n-1})}}_{\text{no jumps}}\underbrace{\lambda dt_n}_{\text{jump}}\underbrace{e^{-\lambda(t-t_n)}}_{\text{no jumps}}}{\frac{(\lambda t)^n}{n!}e^{-\lambda t}} &= n!\frac{dt_1\cdots dt_n}{t^n} \end{aligned} \tag{1.18}$$

To explain the intermediate step of (1.18), the following picture can be useful:



Note that if $N(t) = 1$, we have that

$$\Pr\{T_1 \in dt_1 \mid N(t) = 1\} = \frac{dt_1}{t}, \qquad 0 < t_1 < t. \tag{1.19}$$

and thus T_1 is uniformly distributed in $(0, t)$.
Note that $\Pr\{T_1 > t\} = \Pr\{N(t) = 0\} = p_0(t) = e^{-\lambda t}$ and it is called the "survival probability".

(ii) The mean and variance of the Poisson process are[1]

$$\begin{cases} \mathbb{E}[N(t)] = \lambda t \\ \mathbb{V}ar[N(t)] = \lambda t \end{cases} \tag{1.20}$$

In fact

$$\begin{aligned} \mathbb{E}[N(t)] &= \sum_{h=0}^{\infty} h e^{-\lambda t} \frac{(\lambda t)^h}{h!} = \sum_{h=1}^{\infty} h e^{-\lambda t} \frac{(\lambda t)^h}{h!} = e^{-\lambda t} \sum_{h=1}^{\infty} \frac{(\lambda t)^h}{(h-1)!} \\ &= e^{-\lambda t} \lambda t \sum_{h=1}^{\infty} \frac{(\lambda t)^{h-1}}{(h-1)!} = e^{-\lambda t} \lambda t \sum_{r=0}^{\infty} \frac{(\lambda t)^r}{r!} = e^{-\lambda t} \lambda t e^{\lambda t} = \lambda t \end{aligned} \tag{1.21}$$

and

$$\begin{aligned} \mathbb{E}[N(t)^2] &= \sum_{h=0}^{\infty} h^2 e^{-\lambda t} \frac{(\lambda t)^h}{h!} = e^{-\lambda t} \sum_{h=1}^{\infty} h \frac{(\lambda t)^h}{(h-1)!} = e^{-\lambda t} \lambda t \sum_{r=0}^{\infty} (r+1) \frac{(\lambda t)^r}{r!} \\ &= e^{-\lambda t} \lambda t \left[\sum_{r=0}^{\infty} r \frac{(\lambda t)^r}{r!} + e^{\lambda t} \right] = e^{-\lambda t} \lambda t \left[e^{\lambda t} \sum_{r=0}^{\infty} r e^{-\lambda t} \frac{(\lambda t)^r}{r!} + e^{\lambda t} \right] = e^{-\lambda t} \lambda t \left[e^{\lambda t} \lambda t + e^{\lambda t} \right] \\ &= \lambda t (\lambda t + 1), \end{aligned} \tag{1.22}$$

hence

$$\mathbb{V}ar[N(t)] = (\lambda t)^2 + \lambda t - (\lambda t)^2 = \lambda t \tag{1.23}$$

(iii) The covariance is for $s < t$

$$\begin{aligned} \mathbb{C}ov[N(s), N(t)] &= \mathbb{E}[N(s)N(t)] - \mathbb{E}[N(s)]\mathbb{E}[N(t)] \\ &= \mathbb{E}[N(s)(N(t) - N(s) + N(s))] - \mathbb{E}[N(s)\mathbb{E}[N(t)]] \\ &= \mathbb{E}[N^2(s)] + \mathbb{E}[N(s)(N(t) - N(s))] - \mathbb{E}[N(s)]\mathbb{E}[N(t)] \\ &= \lambda s + \lambda^2 s^2 + \lambda s \lambda (t - s) - \lambda s \lambda t \\ &= \lambda s = \lambda (s \wedge t) \end{aligned} \tag{1.24}$$

(iv) The first-passage time at level k is

$$T_k = \inf\{s : N(s) = k\} \tag{1.25}$$

with distribution

$$\begin{aligned} \Pr\{T_k \in ds\} = \Pr\{N(s) = k-1, N(s, s+ds] = 1\} &= e^{-\lambda s} \frac{(\lambda s)^{k-1}}{(k-1)!} \lambda ds \\ &= \frac{\lambda^k}{\Gamma(k)} e^{-\lambda} s^{k-1} ds \end{aligned} \tag{1.26}$$

[1] Often empirical counting data, viceversa, exhibit significant *overdispersion* (possibily, *underdispersion*), i.e. an index of dispersion $D = \frac{\sigma^2}{\mu} > 1$ ($D < 1$): in this case the fit to a parametric model is improved by the adoption of so called *overdispersed* (*underdispersed*) distributions, i.e. distributions such that the variance exceeds (is lower than) the mean.

and

$$
\begin{aligned}
\Pr\{T_k < \infty\} &= \int_0^\infty e^{-\lambda s}\frac{(\lambda s)^{k-1}}{(k-1)!}\lambda ds = \frac{\lambda^k}{(k-1)!}\int_0^\infty s^{k-1}e^{-\lambda s}ds \\
&= \text{using the change of variable } w=\lambda s \\
= \frac{\lambda^k}{(k-1)!}\int_0^\infty (\frac{w}{\lambda})^{k-1}e^{-w}\frac{dw}{\lambda} &= \frac{1}{(k-1)!}\int_0^\infty w^{k-1}e^{-w}dw = \frac{1}{(k-1)!}\Gamma(k) \\
&= 1.
\end{aligned} \tag{1.27}
$$

This means that the Poisson process reaches any level k in finite time.

(v) The Poisson process $N(t)$ has stationary increments in the sense that the distribution of $N(t)-N(s)$, $t>s>0$ depends only on $(t-s)$

$$
Pr\{N(t)-N(s)=k\} = e^{-\lambda(t-s)}\frac{\lambda^k(t-s)^k}{k!} \tag{1.28}
$$

Proof.

$$
\begin{aligned}
\Pr\{N(t)=k\} &= e^{-\lambda t}\frac{(\lambda t)^k}{k!} \\
= \Pr\{N(t)-N(s)+N(s)=k\} &= \sum_{h=0}^{k}\Pr\{N(t)-N(s)=k-h, N(s)=h\} \\
= \text{(by independence of the increments)} &\sum_{h=0}^{k}\Pr\{N(t)-N(s)=k-h\}\Pr\{N(s)=h\} \\
&= \sum_{h=0}^{k} e^{-\lambda s}\frac{(\lambda s)^h}{h!}\Pr\{N(t)-N(s)=k-h\}
\end{aligned} \tag{1.29}
$$

and the unique solution of the previous equation is (1.28). In fact

$$
\begin{aligned}
\sum_{h=0}^{k} e^{-\lambda s}\frac{(\lambda s)^h}{h!}e^{-\lambda(t-s)}\frac{\lambda^{k-h}(t-s)^{k-h}}{(k-h)!} &= e^{-\lambda t}\frac{\lambda^k}{k!}\sum_{h=0}^{k} k!\frac{s^h(t-s)^{k-h}}{h!(k-h)!} \\
&= e^{-\lambda t}\frac{\lambda^k}{k!}[s+t-s]^k = e^{-\lambda t}\frac{(\lambda t)^k}{k!}
\end{aligned} \tag{1.30}
$$

(vi) The following property relates the Poisson *process* with the binomial *distribution*

$$
\Pr\{N(0,s]=r \mid N(0,t]=k\} = \binom{k}{r}\left(\frac{s}{t}\right)^r\left(1-\frac{s}{t}\right)^{k-r} \qquad 0\le r\le k,\ 0<s<t \tag{1.31}
$$

By taking into account the independence and the homogeneity of the increments of $N(t)$

(property 5), we can write

$$
\begin{aligned}
\Pr\{N(0,s]=r \mid N(0,t]=k\} &= \frac{\Pr\{N(0,s]=r, N(0,t]=k\}}{\Pr\{N(0,t]=k\}} \\
= \frac{\Pr\{N(0,s]=r, N(s,t]=k-r\}}{\Pr\{N(0,t]=k\}} &= \frac{\Pr\{N(0,s]=r\}\Pr\{N(s,t]=k-r\}}{\Pr\{N(0,t]=k\}} \\
&= \frac{e^{-\lambda s}\frac{(\lambda s)^r}{r!}e^{-\lambda(t-s)}\frac{[\lambda(t-s)]^{k-r}}{(k-r)!}}{e^{-\lambda t}\frac{(\lambda t)^k}{k!}}
\end{aligned}
\tag{1.32}
$$

From (1.32), result (1.31) immediately follows.

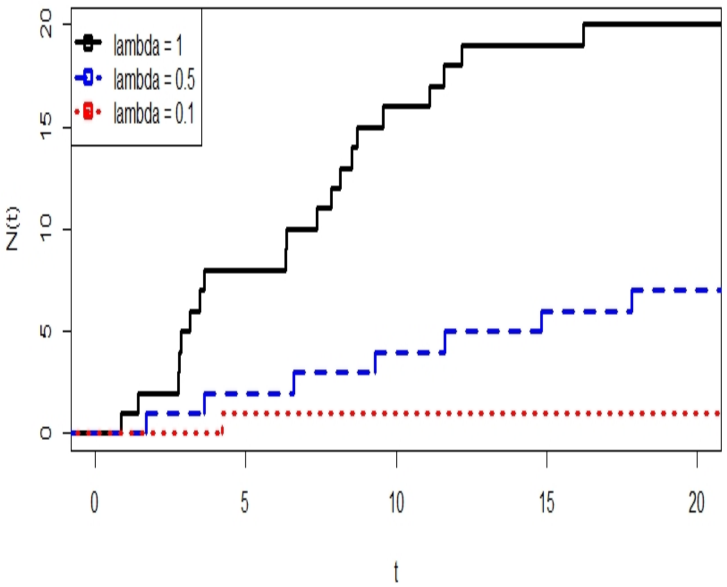


Figure 1.1: Some homogeneous Poisson paths for $\lambda = 1$, $\lambda = 0.5$ and $\lambda = 0.1$.

(vii) By independence, the distribution of the sum of n independent Poisson r.v.'s is obtained by the product of the p.g.f., which in the case of the Poisson distribution is $G_{\mathrm{Poi}(\lambda)}(u) = e^{\lambda(u-1)}$. Therefore given n Poisson r.v.'s $X_1, ..., X_n$ with parameters $\lambda_1, \lambda_2, ..., \lambda_n$, the p.g.f. of the sum $Y = X_1 + X_2 + ... + X_n$ is

$$G_Y(u) = \prod_{i=1}^{n} G_{X_i}(u) = \prod_{i=1}^{n} e^{\lambda_i(u-1)} = e^{\sum_{i=1}^{n}\lambda_i(u-1)} = e^{\Lambda(u-1)} \tag{1.33}$$

where $\Lambda = \sum_{i=1}^{n} \lambda_i$.

The sum of n Poisson r..v.'s is Poisson with parameter $\Lambda = \lambda_1 + \lambda_2 + ... + \lambda_n$ equal to the sum of the single parameters.

1.1.2 Alternative definitions of the Poisson process

An alternative, equivalent definition of the homogeneous Poisson process is the following

Definition 2

1*. $Pr\{N(0)=0\}=1$
2*. $Pr\{N(t)=k\}=e^{-\lambda t}\frac{(\lambda t)^k}{k!}$
3*. *The process has stationary, independent increments.*

It is equivalent to **Definition 1** because:

$$\begin{aligned} Pr\{N(t,t+dt]=0\} &= Pr\{N(t+dt)-N(t)=0\} \\ &= e^{-\lambda t}\frac{(\lambda dt)^0}{0!} = e^{-\lambda t} = 1-\lambda dt + o(dt) \\ Pr\{N(t,t+dt]=1\} &= e^{-\lambda dt}\frac{(\lambda dt)^1}{1!} = \lambda dt e^{-\lambda dt} \\ &= \lambda dt(1-\lambda dt+o(dt)) = \lambda dt + o(dt) \end{aligned} \tag{1.34}$$

A third equivalent definition makes use of the concept of interarrival times.
Let T_k be the time of the k-th event ($T_0=0$). We define the n-th interarrival time τ_n as

$$\tau_n \equiv T_n - T_{n-1} \tag{1.35}$$

so that $\tau_1 = T_1$ and $T_n = \sum_{k=1}^n \tau_k$.
Definition 3.

1**. $Pr\{N(0)=0\}=1$
2**. *The interarrival times* τ_n *are i.i.d.*
3**. *They have exponential distribution, i.e.* $Pr\{\tau_n \le t\} = 1-e^{-\lambda t} \quad \forall n$

Note the no-memory property of the exponential distribution.

$$Pr\{\tau > t+s \mid \tau > s\} = \frac{Pr\{\tau > t+s, \tau > s\}}{Pr\{\tau > s\}} = \frac{Pr\{\tau > t+s\}}{Pr\{\tau > s\}} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda t} = Pr\{\tau > t\}. \tag{1.36}$$

Remark 1.4. Assuming a distribution different from the exponential we obtain a generalised process called renewal process.

From **Definition 3** we have **Definition 1**

$$\begin{aligned} Pr\{N(t,t+dt]=1\} &= Pr\{\tau \le t+dt \mid \tau > t\} \\ &= Pr\{\tau \le dt\} = 1-e^{-\lambda dt} = \lambda dt + o(dt) \\ Pr\{N(t,t+dt]=0\} &= Pr\{\tau > t+dt \mid \tau > t\} = Pr\{\tau > dt\} \\ &= e^{-\lambda dt} = 1-\lambda dt + o(dt) \end{aligned} \tag{1.37}$$

1.2 The weighted Poisson process of order i

The Poisson process of order i, $N^{(i)}$, was introduced in 1984 and 1986 papers [5], [19].
The Poisson process of order i is also defined as a *weighted* sum of independent Poisson processes, i.e.

$$N^{(i)}(t) = \sum_{j=1}^{i} jN_j(t) = \sum_{j=1}^{i}\sum_{h=j}^{i} N_h(t). \tag{1.38}$$

where the processes $N_j(t)$ are i.i.d. Poisson processes, each weighted by j, and the process represents a flow of grouped events where arrivals are "grouped in packages".

The process (1.38) is also studied in [11].
Note that the processes are often assumed identically distributed, i.e. the arrival rate λ is unique. The process can represent situations of multiple claims arrival or collective risks, possibly with heterogeneous (and decreasing) arrival rates $\lambda_1 > \lambda_2 > ... > \lambda_i$.
Its probability generating function derives from a representation (cf. [6]) of the process as a compound Poisson process with discrete-Uniform compounding distribution, i.e. $N^{(i)}(t) = X_1 + X_2 + ... + X_{N_{i\lambda}}(t)$, where $N_{i\lambda}(t)$ is a Poisson process of rate $i\lambda$, and X_h are Uniform r.v.'s over the set of the first i integers.[2]
Hence, it is obtained as

$$\begin{aligned} G^{(i)}(u,t) = \mathbb{E}\left[u^{N^{(i)}(t)}\right] &= \mathbb{E}\left[u^{\sum_{h=0}^{N_{i\lambda}(t)} X_h}\right] \\ = \mathbb{E}\left[\mathbb{E}\left[u^{\sum_{h=0}^{N_{i\lambda}(t)}} | N_{i\lambda}(t)\right]\right] &= \mathbb{E}\left[\mathbb{E}\left(u^{X_h}\right)^{N_{i\lambda}(t)}\right] \\ = \sum_{h=0}^{\infty} \mathbb{E}\left[u^{X_h}\right]^h \cdot e^{-i\lambda t}\frac{(i\lambda t)^h}{h!} &= e^{-i\lambda t} e^{i\lambda t \cdot \mathbb{E}u^X} \\ &= e^{-i\lambda t(1-\psi_X(u))} \end{aligned} \tag{1.39}$$

where the p.g.f. of the discrete-Uniform is $\psi_X(u) = \frac{u}{i}\frac{1-u^i}{1-u}$[3].
An explicit expression of the state probabilities can be found by the coefficients of the p.g.f. (cf. [10] for its expression for $k > i$)

$$\begin{cases} p_0^{(i)}(t) = e^{-i\lambda t} \\ p_k^{(i)}(t) = e^{-i\lambda t}\sum_{h=1}^{k}\binom{k-1}{h-1}\frac{(\lambda t)^h}{h!} \qquad k = 1, 2, ..., i \end{cases} \tag{1.40}$$

One can note that the maximum number of events for a given i over an interval dt corresponds to

[2] Exploiting the discrete type distribution of the compound Poisson-discrete-Uniform representation, [10] provides a Panjer-type recursion of the probability mass function:

$$\begin{cases} p_1^{(i)}(t) = \lambda t p_0^{(i)}(t) \\ p_k^{(i)}(t) = (2 + \frac{\lambda t - 2}{k})p_{k-1}^{(i)}(t) - (1 - \frac{2}{k})p_{k-2}^{(i)}(t) \qquad k = 2, 3, ..., i \\ p_k^{(i)}(t) = (2 + \frac{\lambda t - 2}{k})p_{k-1}^{(i)}(t) - (1 - \frac{2}{k})p_{k-2}^{(i)}(t) - \frac{i+1}{k}\lambda t p_{k-i-1} + \frac{i}{k}\lambda t p_{k-i-2} \qquad k \geq i+1 \end{cases}$$

where, as above, $p_0^{(i)}(t) = e^{-i\lambda t}$.

[3] The p.g.f. of the discrete-Uniform r.v.'s X_h, given that the probability mass function of the discrete-Uniform distribution over the set of the first i integers is $\frac{1}{i}$, is obtained as $\mathbb{E}[u^{X_h}] = \sum_{k=1}^{i} u^k \frac{1}{i} = \frac{u}{i}\frac{1-u^i}{1-u}$.

the sum of the first i integers, i.e. the maximal jump size is $\frac{i(i+1)}{2}$, which can be substantially larger than 1.
Its mean and variance (cf. Fig. 1.2 and Fig. 1.3 resp.) follow the expression for the sum of the first i integers, namely

$$\begin{cases} \mathbb{E}[N^{(i)}(t)] = \lambda t \sum_{j=1}^{i} j = \frac{i(i+1)}{2}\lambda t \\ \mathbb{V}[N^{(i)}(t)] = \lambda t \sum_{j=1}^{i} j^2 = \frac{i(i+1)(2i+1)}{6}\lambda t \end{cases} \tag{1.41}$$

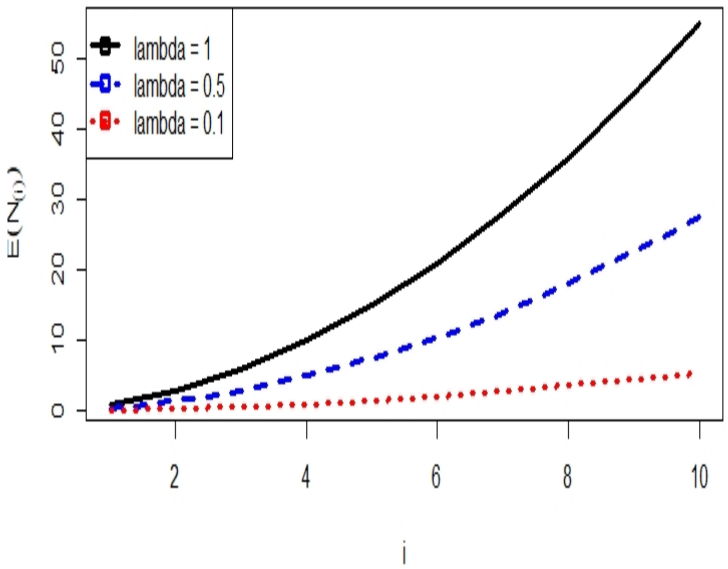


Figure 1.2: Mean of the weighted Poisson process w.r.t. the process index i ($t = 1$).

The process is overdispersed, that is the ratio of variance over the mean, also known as Fisher index is greater than one, i.e. $FI = 1 + \frac{2}{3}(i-1) > 1$, which in turn increases with the order i of the process.
A second representation is described in [6], where the process is represented as a pure-birth representation of the weighted Poisson process, where

$$\begin{cases} \Pr\{N^{(i)}(t+dt) = h+k | N^{(i)}(t) = h\} = \lambda dt + o(dt) \qquad k = 1, 2, ..., i \text{ (multiple jumps)} \\ \Pr\{N^{(i)}(t+dt) = h+k | N^{(i)}(t) = h\} = 1 - i\lambda dt + o(dt) \qquad i = 0 \end{cases} \tag{1.42}$$

and implicitly, for jumps of amplitude greater than i, $\Pr\{N^{(i)}(dt) \geq i+1\} \sim o(dt)$.
Alternatively, a more compact notation is (see [11])

$$\begin{cases} \Pr\{dN^{(i)}(t) = h\} = \lambda dt \qquad 1 \leq h \leq i \\ \Pr\{dN^{(i)}(t) = 0\} = 1 - i\lambda dt \end{cases} \tag{1.43}$$

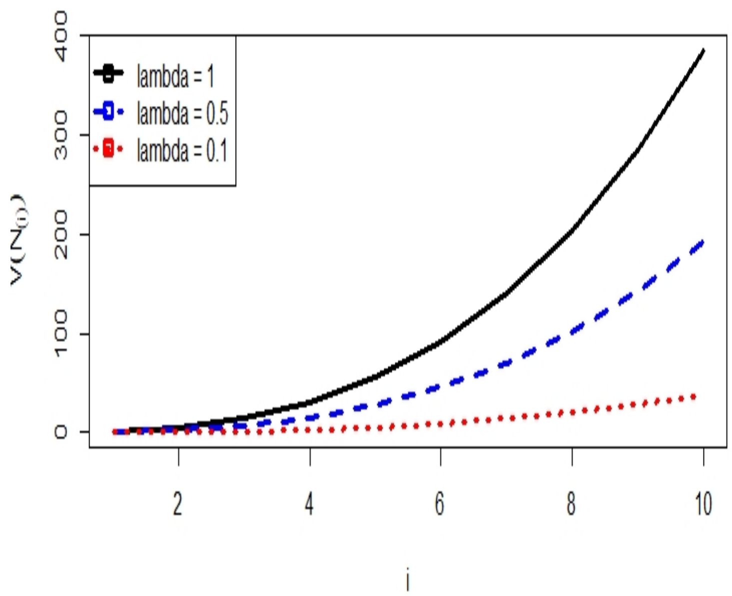


Figure 1.3: Variance of the weighted Poisson process w.r.t. the process index i ($t = 1$).

1.3 Skellam process: the difference between two Poisson processes

The process obtained as the difference of two Poisson processes was studied first in 1946, by Skellam. It is defined as $S(t) = N_1(t) - N_2(t)$, where $N_1(t)$, $N_2(t)$ are independent Poisson processes, $t > 0$, with rates $\lambda_1 > 0$, $\lambda_2 > 0$, and take integer values (i.e., both positive or negative). Jumps have unitary size (for a Skellam process with multiple, "simultaneous" jumps of arbitrary size).
The probability mass function of $S(t)$ turns out to be of the form

$$p_k(t) \equiv \Pr\{S(t) = k\} = e^{-(\lambda_1+\lambda_2)t}(\frac{\lambda_1}{\lambda_2})^{\frac{k}{2}} I_{|k|}(2\sqrt{\lambda_1\lambda_2}t), \quad k \in \mathbb{Z} \equiv \{0, \pm 1, \pm 2, \pm 3, ...\} \tag{1.44}$$

where $I_k(x)$ is the modified Bessel function of the first kind, defined as $I_k(x) = \sum_{n=0}^{\infty} \frac{(\frac{x}{2})^{2n+k}}{n!\Gamma(n+k+1)}$ (cf. Fig. 1.4) .Of course, if $\lambda_1 = \lambda_2 = \lambda$ the distribution $\{p_k, k \geq 0\}$ is symmetric and reads $p_k(t) = e^{-2\lambda t} I_{|k|}(2\lambda t)$; positive (negative) skewness corresponds to respectively $\lambda_1 > \lambda_2$ ($\lambda_1 < \lambda_2$). The mean, variance, covariance and p.g.f. of the distribution are respectively

$$\begin{cases} \mathbb{E}[S(t)] = (\lambda_1 - \lambda_2)t, \\ \mathbb{V}ar[S(t)] = (\lambda_1 + \lambda_2)t, \\ \mathbb{C}ov(S(t), S(s)) = (\lambda_1 + \lambda_2)\min(s,t), \\ G_S(u) = G_{N_1(u)}(u) G_{N_2}(\frac{1}{u}) \end{cases} \tag{1.45}$$

-

Note that the process is oversdispersed.

Theorem 1.4: By the linearity of the differential operator, the state transition law is governed

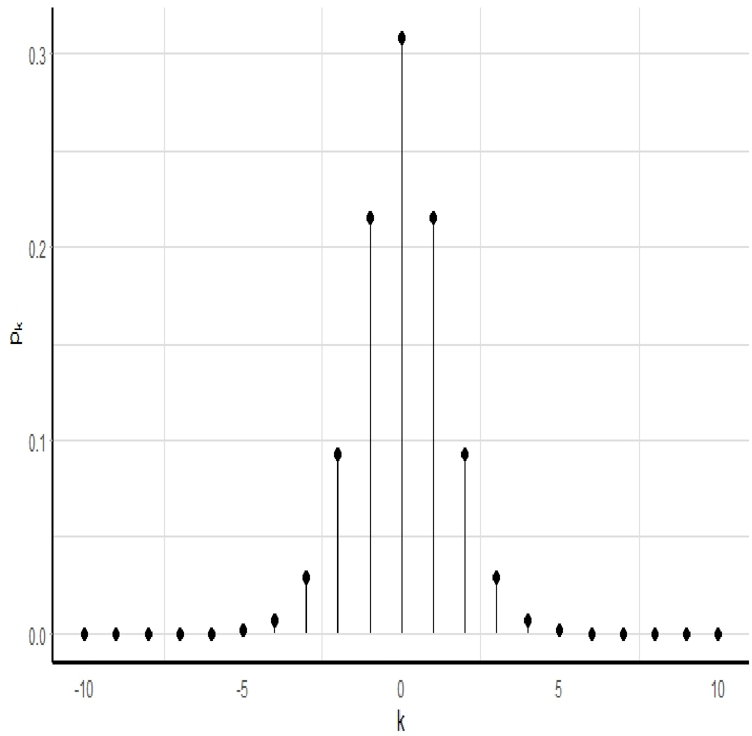


Figure 1.4: Probability mass functions of the Skellam process with symmetry ($\lambda_1 = \lambda_2 = 1$) w.r.t. ($t = 1$).

by the differential equation

$$\frac{d}{dt}p_k(t) \equiv -(\lambda_1 p_k(t) - \lambda_1 p_{k-1}(t)) + (\lambda_2 p_{k+1}(t) - \lambda_2 p_k(t)) \tag{1.46}$$

with initial conditions $p_0(0) = 1$, $p_k(0) = 0$.

Proof. During an infinitesimal interval of time of length dt the state probability $p_k(t)$ satisfies the following relationship, i.e.

$$\begin{aligned} Pr\{S(t+dt) = k\} &\equiv p_k(t+dt) \\ = p_k(t)(1-\lambda_1 dt)(1-\lambda_2 dt) + p_{k-1}(t)\lambda_1 dt(1-\lambda_2 dt) &+ p_{k+1}(t)(1-\lambda_1 dt)\lambda_2 dt \\ = p_k(t) - p_k(t)\lambda_1 dt - p_k(t)\lambda_2 dt &+ p_{k-1}(t)\lambda_1 dt + p_{k+1}(t)\lambda_2 dt \end{aligned} \tag{1.47}$$

and after rearranging terms and taking the limit for $dt \to 0$ the equation

$$\frac{d}{dt}p_k(t) = -p_k(t)(\lambda_1 + \lambda_2) + \lambda_1 p_{k-1}(t) + \lambda_2 p_{k+1}(t) \tag{1.48}$$

is obtained.

The sample paths of the Skellam process have unitary upwards and downwards jumps: as such the Skellam distribution is suited for modelling data representing differences (e.g., soccer scorings) or increments (e.g., up and down price movements). An actuarial example is the application to experience (bonus-malus) rating in motor-vehicle insurance.

The moment generating function of the Skellam process reads as (see [4])

$$M(u;t) = \mathbb{E}[e^{uS}] = e^{-(\lambda_1(e^u-1)+\lambda_2(e^u-1))t} \qquad |u| \leq 1 \tag{1.49}$$

Remark 1.5. The Skellam process is a special case of a birth-death process with $\lambda_k \equiv \lambda_1$ and $\mu_k \equiv \lambda_2$. See paragraph 4.

1.4 The non-homogeneous Poisson process

A generalisation of the Poisson process can be obtained by assuming that the rate function $\lambda = \lambda(t)$ depends on time t (cf. Fig. 1.5).
In this situation the definition of an homogeneous Poisson process can be rewritten as:

1. $\Pr\{N(0) = 0\} = 1$;
2. $\Pr\{N(t, t + dt] = 1\} = \lambda(t)dt + o(dt)$, where $\lambda(t) > 0$, $t > 0$;
3. $\Pr\{N(t, t + dt] = 0\} = 1 - \lambda(t)dt + o(dt)$;
4. $\Pr\{N(t, t + dt] > 1\} = o(dt)$;
5. If $0 = t_0 < t_1 < \ldots < t_n < t$, the random variables $N(t_j) - N(t_{j-1})$, $1 \le j \le n$, are independent,

so that equation (1.1) must be replaced by:

$$\frac{d}{dt}p_k(t) = -\lambda(t)p_k(t) + \lambda(t)p_{k-1}(t), \qquad k \ge 0, \tag{1.50}$$

with the initial conditions

$$p_k(0) = \begin{cases} 1 & k = 0 \\ 0 & k > 0 \end{cases} \tag{1.51}$$

and $p_{-1}(t) = 0$.
The probability generating function $G(u, t)$, $|u| \le 1$, $t > 0$, emerges by solving the following partial

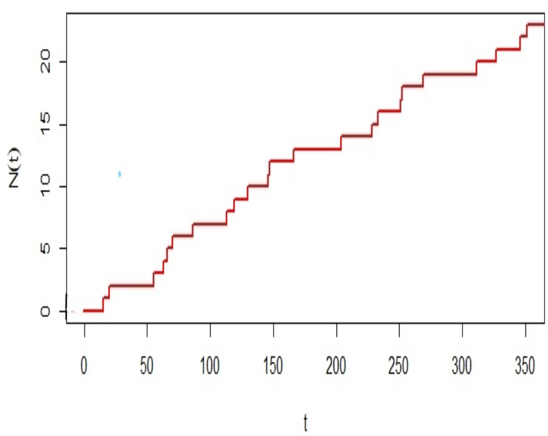


Figure 1.5: A typical sample path of the non-homogeneous Poisson path with non-linear (decreasing and logistic) intensity (i.e., $\lambda(t) = \frac{r(C-n_0)}{C-n_0+n_0e^{rt}}$, where C, n_0 and r are respectively the maximum point or capacity of the system, the minimum point and the growth rate of a logistic growth).

differential equation:

$$\begin{cases} \frac{\partial}{\partial t} G(u,t) = \lambda(t)(u-1)G(u,t), \\ G(u,0) = 1. \end{cases} \tag{1.52}$$

From (1.52) it is clear that

$$G(u,t) = e^{-(1-u)\int_0^t \lambda(s)ds} = e^{-\int_0^t \lambda(s)ds} \left[\sum_{k=0}^{\infty} u^k \frac{\left[\int_0^t \lambda(s)ds\right]^k}{k!} \right], \tag{1.53}$$

so that the probability distribution of the non-homogeneous Poisson process becomes

$$p_k(t) = e^{-\int_0^t \lambda(s)ds} \frac{\left[\int_0^t \lambda(s)ds\right]^k}{k!}, \qquad k \geq 0. \tag{1.54}$$

1.5 Compound Poisson processes

Let $N(t)$ be the number of claims generated by an insurance policy in a fixed time interval $[0,t]$ and let X_j, $j \geq 1$, be i.i.d. r.v.'s and also independent from the random number $N(t)$. If X_j is the claim amount related to the j-th accident, the total amount of claims generated by this policy, in the given fixed time period $[0,t]$, is:

$$Z(t) = \sum_{j=1}^{N(t)} X_j. \tag{1.55}$$

Theorem 1.5: The mean and variance of $Z(t)$ are:

$$\begin{aligned} \mathbb{E}[Z(t)] &= \mathbb{E}[N(t)]\,\mathbb{E}(X) \\ \mathbb{V}\mathrm{ar}[Z(t)] &= \mathbb{E}[X]^2 \mathbb{V}\mathrm{ar}[N(t)] + \mathbb{E}[N(t)]\mathbb{V}\mathrm{ar}(X) = \mathbb{V}\mathrm{ar}[N(t)][\mathbb{E}^2(X) + \mathbb{V}\mathrm{ar}(X)] = \mathbb{V}\mathrm{ar}[N(t)]\mathbb{E}[X^2]. \end{aligned} \tag{1.56}$$

Proof. For the mean

$$\mathbb{E}[Z(t)] = \mathbb{E}[\mathbb{E}[Z(t) \mid N(t)]] = \mathbb{E}[\mathbb{E}[\sum_{j=1}^{N} X_j \mid N(t)]] = \mathbb{E}[N(t)\mathbb{E}[X \mid N(t)]] = \mathbb{E}[N(t)]\mathbb{E}[X] \tag{1.57}$$

For the variance we use the following
Lemma 1: For arbitrary r.v.'s X, Y the following decomposition holds,

$$\mathbb{V}ar[Z] = \mathbb{E}_Y[\mathbb{V}ar[Z \mid Y]] + \mathbb{V}ar_Y[\mathbb{E}[Z \mid Y]]. \tag{1.58}$$

Proof Lemma 1.

$$\begin{aligned} \mathbb{V}\mathrm{ar}[Z] = \mathbb{E}[(Z - \mathbb{E}[Z])^2 = \mathbb{E}[(Z - E[Z \mid Y] + E[Z \mid Y] - E[Z])^2] \\ = \mathbb{E}_Y[\mathbb{E}[(Z - \mathbb{E}[Z \mid Y])^2 + (\mathbb{E}[Z \mid Y] - \mathbb{E}[Z])^2 \mid Y] + 2\mathbb{E}_Y[\mathbb{E}[(Z - \mathbb{E}[Z \mid Y])(\mathbb{E}[Z \mid Y] - \mathbb{E}[Z]) \mid Y]] \end{aligned} \tag{1.59}$$

but the second component is zero being

$$2(\mathbb{E}[Z \mid Y] - \mathbb{E}[Z]) \cdot \mathbb{E}[Z - E[Z \mid Y] \mid Y] = 0. \tag{1.60}$$

Moreover

$$\mathbb{E}_Y[\mathbb{E}[(Z - \mathbb{E}[Z \mid Y])^2 \mid Y]] = \mathbb{E}_Y[\mathbb{V}ar[Z \mid Y]] \tag{1.61}$$

and

$$\mathbb{E}_Y[\mathbb{E}[\mathbb{E}[Z \mid Y] - \mathbb{E}[Z]]^2 \mid Y] = \mathbb{E}[\mathbb{E}[Z \mid Y] - \mathbb{E}[Z]]^2 = \mathbb{V}ar_Y[\mathbb{E}[Z \mid Y]] \tag{1.62}$$

so that $\mathbb{V}ar[Z] = \mathbb{V}ar_Y[\mathbb{E}[Z \mid Y]] + \mathbb{E}_Y[\mathbb{V}ar[Z \mid Y]]$.
The variance is decomposed in the variance of the conditional mean plus the mean of the conditional variance.
Applying Lemma 1 to the Theorem 1.5

$$\mathbb{V}ar[Z(t)] = \mathbb{V}ar_N[\mathbb{E}[Z(t) \mid N(t)]] + \mathbb{E}_N[\mathbb{V}ar[Z(t) \mid N(t)]] \tag{1.63}$$

so that

$$\begin{cases} \mathbb{V}ar_N[\mathbb{E}[Z(t) \mid N(t)]] = \mathbb{V}ar_N[N(t)\mathbb{E}[X]] = \mathbb{E}[X]^2\mathbb{V}ar[N(t)], \\ \mathbb{E}_N[\mathbb{V}ar[Z(t) \mid N(t)]] = \mathbb{E}_N[N(t)\mathbb{V}ar[X]] = \mathbb{E}[N(t)]\mathbb{V}ar[X]. \end{cases} \tag{1.64}$$

We used the property of the conditional expectation $\mathbb{E}_Y[\mathbb{E}[Z \mid Y]] = \mathbb{E}[Z]$.

In particular, if $Z = 1_A$ is the indicator function of event A,

$$\mathbb{E}_Y[Pr\{A \mid Y\}] = Pr\{A\} = \mathbb{E}[Z] \qquad \square \tag{1.65}$$

If $N(t)$ is Poisson and the X_j are non-negative i.i.d. r.v.'s with probability distribution function F we have that the probability generating function of the compound Poisson process is:

$$\mathbb{E}\left[u^{Z(t)}\right] = e^{-\lambda t + \lambda t \mathbb{E}\left[u^X\right]} = e^{-\lambda t \int_0^\infty (1-u^x) dF(x)}. \tag{1.66}$$

In fact if the r.v.'s X_i are i.i.d. Poisson r.v.'s, then

$$\begin{aligned} \mathbb{E}\left[u^{Z(t)}\right] = \mathbb{E}\left[u^{\sum_{j=1}^{N(t)} X_j}\right] = \mathbb{E}_N\left[\mathbb{E}\left[u^{\sum_{j=1}^{N(t)} X_j} | N(t)\right]\right] = \mathbb{E}_N\left[\mathbb{E}\left[u^X\right]^{N(t)}\right] \\ \sum_{k=0}^{\infty} \mathbb{E}\left[u^X\right]^k \cdot p_k(t) = \sum_{k=0}^{\infty} \mathbb{E}\left[u^X\right]^k \cdot e^{-\lambda t} \frac{(\lambda t)^k}{k!} = e^{-\lambda t} e^{\lambda t \cdot \mathbb{E}[u^X]} = e^{-\lambda t} e^{\lambda t \cdot \int_0^\infty u^x dF(x)} \\ = e^{-\lambda t \{1 - \int_0^\infty u^x dF(x)\}} = e^{-\lambda t \int_0^\infty (1-u^x) dF(x)}. \end{aligned} \tag{1.67}$$

and the result obtains. $\square$

Remark 1.6. If $X_j = N_j(t)$ the process $\sum_{j=1}^{N(t)} N_j(t)$ is called compound Poisson-Poisson. It represents the Poisson sum of Poisson processes.

Remark 1.7. For a portfolio of Q policies the total aggregate claim is $Z(t) = \sum_{q=1}^{Q} \sum_{j=1}^{N_q(t)} X_{q,j}$, where the $X_{q,j}$'s are i.i.d. Hence mean, variance and p.g.f. are given respectively as

$$\mathbb{E}[Z(t)] = \textstyle\sum_q \mathbb{E}[N_q(t)]\mathbb{E}[X_q],$$
$$\mathbb{V}\mathrm{ar}[Z(t)] = \textstyle\sum_q \mathbb{V}\mathrm{ar}[N_q(t)] \cdot \left[\mathbb{E}^2(X_q) + \mathbb{V}\mathrm{ar}(X_q)\right],$$
$$\mathbb{E}\left[u^{\sum_q Z_q(t)}\right] = e^{-t\sum_q \lambda_q + t\sum_q \lambda_q \mathbb{E}u^{X_q}} = e^{-t\sum_q \lambda_q(1-\mathbb{E}u^{X_q})} = e^{-t\sum_q \lambda_q \int_0^\infty (1-u^x)dF_q(x)}.$$

If to each Poisson event (a car accident) we associate a r.v. X_j (the amount of the damage) it is of interest to calculate the following probability:

$$\Pr\left\{\max\left(X_1, \dots, X_{N(t)}\right) < b\right\} \tag{1.68}$$

which represents the maximal damage produced by the $N(t)$ events in the interval $(0, t)$.

Note that for $N(t) = 0$ we assume that $\max\left(X_1, \dots, X_{N(t)}\right) = 0$.
The probability (1.68) can be calculated as

$$\begin{aligned}
\Pr\left\{\max\left(X_1, \dots, X_{N(t)}\right) < b\right\} &= \mathbb{E}\left[\Pr\left\{\max\left(X_1, \dots, X_{N(t)}\right) < b \mid N(t)\right\}\right] \\
&= \sum_{k=0}^{\infty} \Pr\left\{\max\left(X_1, \dots, X_k\right) < b\right\} \Pr\{N(t) = k\} \\
&= \sum_{k=0}^{\infty} [\Pr(X < b)]^k e^{-\lambda t} \frac{(\lambda t)^k}{k!} \\
&= e^{-\lambda t + \lambda t \Pr\{X < b\}}.
\end{aligned} \tag{1.69}$$

Note that the distribution function of the random variable $\max\left(X_1, \dots, X_{N(t)}\right)$ reads

$$\Pr\left\{\max\left(X_1, \dots, X_{N(t)}\right) < b\right\} = \begin{cases} 0 & b < 0 \\ e^{-\lambda t} & b = 0 \\ e^{-\lambda t + \lambda t \Pr(X < b)} & b > 0 \end{cases} \tag{1.70}$$

and shows a jump of height $e^{-\lambda t}$ at $b = 0$.
In the special case when X is exponentially distributed, that is

$$\Pr(X < b) = \mu \int_0^b e^{-\mu y} dy = 1 - e^{-\mu b}, \tag{1.71}$$

for $b > 0$, the probability distribution (1.70) becomes

$$\Pr\left\{\max\left(X_1, \dots, X_{N(t)}\right) < b\right\} = \begin{cases} 0 & b < 0 \\ e^{-\lambda t} & b = 0 \\ e^{-\lambda t e^{-\mu b}} & b > 0 \end{cases} \tag{1.72}$$

and is known as the Gumbel or Gnedenko distribution.

If we consider the special compound Poisson process

$$\sum_{j=1}^{N(\delta^{-\alpha})} X_j \quad 0<\alpha<1, \delta>0 \tag{1.73}$$

where the X_j are positive i.i.d. r.v.'s with distribution

$$Pr(X>x)=\begin{cases} 1 & x\leq\delta \\ \delta^{\alpha}x^{-\alpha} & x>\delta \end{cases} \tag{1.74}$$

and $N(\delta^{-\alpha})$ is a Poisson r.v. with parameter $\lambda=\delta^{-\alpha}$, $\delta>0$, we have that, for $\delta\to 0$, (1.73) converges in distribution to an α-Stable subordinator $S_\alpha(\sigma,1,0)$ of order $0<\alpha<1$ with characteristic function

$$\mathbb{E}\left[e^{i\theta S_\alpha(\sigma,1,0)}\right]=e^{-|\theta|^\alpha\sigma^\alpha}\left(1-isgn(\theta)tg\left(\frac{\pi}{2}\alpha\right)\right) \tag{1.75}$$

where $\sigma=\{\Gamma(1-\alpha)\cos\left(\frac{\pi}{2}\alpha\right)\}^{\frac{1}{\alpha}}$.
See the Appendix A for the general expression of the characteristic function of an α-Stable r.v. $S_\alpha(\mu,\beta,\sigma)$, when $\mu\neq 0$, $-1<\beta<1$.

1.6 Poisson random fields

Points randomly distributed in the Euclidean space $\mathbb{R}^n$ can be modelled by means of a multi-index Poisson process, called Poisson field.
The basic property of the Poisson random fields (cf. Fig. 1.8) is represented by the independency of the random variables $N(S_A)$ and $N(S_B)$ for arbitrary disjoint sets S_A and S_B such that $S_A\cap S_B=\emptyset$.
We have that for the Poisson random field of rate $\lambda>0$ we have that

$$\Pr\{N(S)=k\}=\frac{\lambda^k[\mu(S)]^k}{k!}e^{-\lambda\mu(S)} \qquad k\geq 0 \tag{1.76}$$

where $\mu(S)$ is the volume of the set $S\subset\mathbb{R}^n$ and $N:\mathcal{B}^n\to\mathbb{N}\cup\{0\}$ when $\mathcal{B}^n$ is the Borel class over $\mathbb{R}^n$.

Remark 1.8. The volume of a sphere S_r^n of radius r in $\mathbb{R}^n$ is

$$\mu(S_r^n)=\int_0^r Area(S_r^n)dr,$$

where

$$\int_0^r Area(S_1^n)\rho^{n-1}d\rho=\int_0^r \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\rho^{n-1}d\rho=\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\int_0^r\rho^{n-1}d\rho=\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\frac{r^n}{n}=\frac{\pi^{\frac{n}{2}}r^n}{\Gamma(\frac{n}{2}+1)}.$$

Remark 1.8 (cont.). For $n=3$,

$$\mu(S_r^3)=\frac{\pi^{\frac{3}{2}}r^3}{\Gamma(\frac{3}{2}+1)}=\frac{4}{3}\frac{\pi^{\frac{3}{2}}r^3}{\sqrt{\pi}}=\frac{4}{3}\pi r^3,$$

where, according to the properties of the Gamma function, i.e. $\Gamma(z+1)=z!=z\Gamma(z)$, $\Gamma(\frac{1}{2})=\sqrt{\pi}$, $1=\Gamma(2)=\Gamma(1)=0!$, $\Gamma(-z)=\pm\infty$ $\forall z$ positive integer

$$\Gamma(\tfrac{3}{2}+1)=\tfrac{3}{2}\Gamma(\tfrac{3}{2})=\tfrac{3}{2}\Gamma(\tfrac{1}{2}+1)=\tfrac{3}{2}\tfrac{1}{2}\Gamma(\tfrac{1}{2})=\tfrac{3}{4}\sqrt{\pi},$$

Moreover

$$Area(S_1^n)=\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$$

where

$$S_r^n=\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}r^{n-1}$$

A proof can be obtained from

$$\int\cdots\int_{\mathbb{R}^n}\frac{e^{-\frac{x_1^2}{2}}}{\sqrt{2\pi}}\cdots\frac{e^{-\frac{x_n^2}{2}}}{\sqrt{2\pi}}dx_1\ldots dx_n=1$$

which is a multinormal distribution with independent marginals.
The above integral can be evaluated by means of hyperspheric coordinates as

$$\int_0^\infty Area(S_1^n)r^{n-1}\frac{e^{-\frac{r^2}{2}}}{(2\pi)^{\frac{n}{2}}}dr=1$$

so that

$$Area(S_1^n)=\frac{(2\pi)^{\frac{n}{2}}}{\int_0^\infty r^{n-1}e^{-\frac{r^2}{2}}\,dr}=\frac{(2\pi)^{\frac{n}{2}}}{2^{\frac{n}{2}-1}\Gamma(\frac{n}{2})}$$

being $\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}dx$ with the change of variable $x=\frac{r^2}{2}$.

For the distribution of the nearest neighbour in the Poisson random field we have that

$$\begin{aligned}&\Pr\{\text{the nearest neighbour to a fixed point with center at } x \text{ is outside the sphere } S_r \text{ of radius r}\}\\&=\Pr\{N(S_r)=0\}=e^{-\lambda\mu(S_r)}.\end{aligned} \tag{1.77}$$

For example, in $\mathbb{R}^2$ and $\mathbb{R}^3$ we have that

$$\Pr\{N(S_r)=0\}=\begin{cases}e^{-\lambda\pi r^2} & \text{in } \mathbb{R}^2\\ e^{-\frac{4}{3}\lambda\pi r^3} & \text{in } \mathbb{R}^3\end{cases} \tag{1.78}$$

For the planar case $\mathbb{R}^2$, the density related to (1.78) reads

$$f(r)=2\lambda\pi re^{-\lambda\pi r^2}\qquad r>0 \tag{1.79}$$

which therefore is a Rayleigh distribution. In $\mathbb{R}^n$ we have

$$\Pr\{N(S_r) = 0\} = \exp\left(-\lambda \frac{\pi^{\frac{n}{2}} r^n}{\Gamma(\frac{n}{2}+1)}\right). \tag{1.80}$$

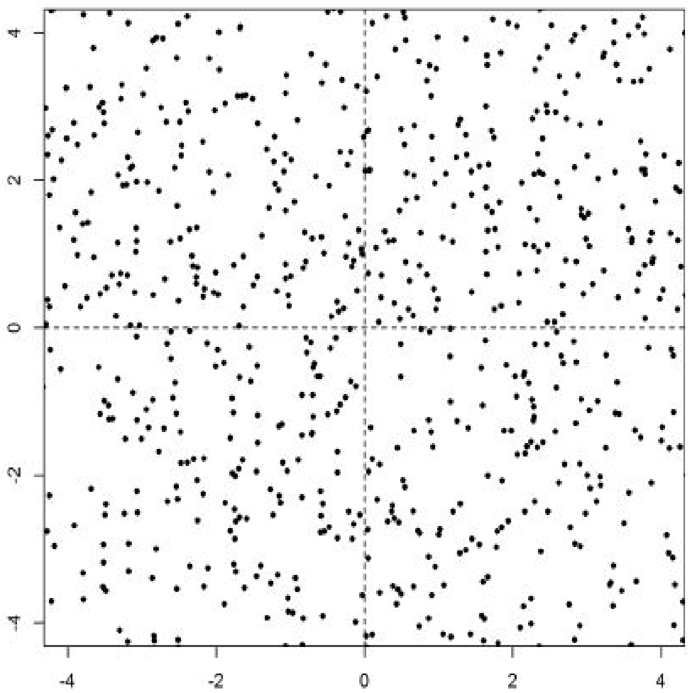


Figure 1.6: A general sample path of a Poisson random field.

1.7 The integral of an homogeneous Poisson process

In this section we study the integral of an homogeneous Poisson process (see [15]):

$$\mathcal{N}(t) = \int_0^t N(s)ds = \sum_{j=2}^{n}(T_j - T_{j-1})(j-1) + n(t - T_n) \tag{1.81}$$

where, as in (1.18),

$$\Pr\{T_1 \in dt_1, \cdots, T_n \in dt_n \mid N(t) = n\} = \frac{n!dt_1 \cdots dt_n}{t^n}, \qquad \text{with } 0 < t_1 < t_2 < \cdots < t_n < t. \tag{1.82}$$

Remark 1.9. The $T_j - T_{j-1}$ are distributed as the difference between order statistics from Uniform$[0,T]$ r.v.'s (see [6], cf. also [7]). *As an example, the distribution of the difference of order statistics is of interest in statistical seismology, in which case the difference of order statistics of magnitudes following a Gutenberg-Richter (i.e., exponential) law is (again) exponential.*

It is possible to show that the conditional characteristic function of $\mathcal{N}(t)$ reads

$$\mathbb{E}\left(e^{i\beta\mathcal{N}(t)}\middle|\mathcal{N}(t) = n\right) = \frac{(e^{i\beta t}-1)^n}{t^n(i\beta)^n} \tag{1.83}$$

while the unconditional characteristic function is

$$\mathbb{E}\left(e^{i\beta\mathcal{N}(t)}\right) = e^{\lambda\int_0^t (e^{i\beta s}-1)ds} \tag{1.84}$$

because

$$\begin{aligned}
\mathbb{E}\left[e^{i\beta\mathcal{N}(t)}\right] &= \mathbb{E}\left[e^{i\beta\int_0^t N(s)ds}\right] = \sum_{n=0}^{\infty}\mathbb{E}\left[e^{i\beta\int_0^t N(s)ds} \mid N(t)=n\right]\cdot Pr\{N(t)=n\} \\
&= \sum_{n=0}^{\infty}\frac{(e^{i\beta t}-1)^n}{t^n(i\beta)^n}e^{-\lambda t}\frac{(\lambda t)^n}{n!} = e^{-\lambda t}\sum_{n=0}^{\infty}\frac{\left[\lambda(e^{i\beta t}-1)\right]^n}{(i\beta)^n n!} \\
&= e^{-\lambda t}e^{\frac{\lambda(e^{i\beta t}-1)}{i\beta}} = e^{-\lambda t}e^{\lambda\int_0^t e^{i\beta s}ds} = e^{\lambda\int_0^t (e^{i\beta s}-1)ds}.
\end{aligned} \tag{1.85}$$

For small values of t the following approximation holds:

$$\mathbb{E}\left(e^{i\beta\mathcal{N}(t)}\right) = e^{i\lambda\beta\frac{t^2}{2}-\lambda\beta^2\frac{t^3}{6}+o(t^3)} \tag{1.86}$$

This implies that, for small value of t, the area below an homogeneous Poisson process is distributed as a normal with mean $\mathbb{E}[\mathcal{N}(t)] = \frac{\lambda t^2}{2}$ and variance $\mathbb{V}\mathrm{ar}[\mathcal{N}(t)] = \frac{\lambda t^3}{3}$, since the c.f. of a Normal distribution $N(\mu,\sigma^2)$ is $\mathbb{E}\left[e^{-i\beta N(\mu,\sigma^2)}\right] = e^{i\beta\mu-\beta^2\frac{\sigma^2}{2}}$.
Moreover

(i) the relationship

$$\mathbb{E}\left\{\int_0^t N^k(s)ds\middle|N(t)=n\right\} = \frac{t}{n+1}\sum_{j=1}^{n} j^k = \begin{cases} \frac{nt}{2} & k=1 \\ \frac{n(2n+1)t}{6} & k=2 \\ \frac{n^2(n+1)t}{4} & k=3 \end{cases} \tag{1.87}$$

holds, where we applied the following formulas

$$\begin{aligned}
\textstyle\sum_{j=1}^n j &= \tfrac{n(n+1)}{2} \\
\textstyle\sum_{j=1}^n j^2 &= \tfrac{n(n+1)(2n+1)}{6} \\
\textstyle\sum_{j=1}^n j^3 &= \left(\tfrac{n(n+1)}{2}\right)^2 = \left(\textstyle\sum_{j=1}^n j\right)^2;
\end{aligned} \tag{1.88}$$

(ii) by conditioning out (1.87) we easily have the unconditional values

$$\mathbb{E}\left\{\int_0^t N(s)^k ds\right\} = \begin{cases} \frac{\lambda t^2}{2} & k=1 \\ \frac{\lambda^2 t^3}{3}+\frac{\lambda t^2}{2} & k=2 \\ \frac{\lambda^3 t^4}{4}+\lambda^2 t^3+\frac{\lambda t^2}{2} & k=3 \end{cases} \tag{1.89}$$

For the second order moment of (1.81) we have the following results

$$\begin{aligned}
\mathbb{E}\left\{\left(\int_0^\infty N(s)ds\right)^2\middle|N(t)=n\right\} &= \tfrac{n(3n+1)t^2}{12} \\
\mathbb{V}ar\left\{\left(\int_0^\infty N(s)ds\right)^2\middle|N(t)=n\right\} &= \tfrac{nt^2}{12}
\end{aligned} \tag{1.90}$$

as stated in [15]. From (1.89) we extract the variance of the integral of the Poisson process

$$\mathbb{V}ar\left(\int_0^t N(s)ds\right) \quad = \quad \mathbb{E}\left(\mathbb{V}ar\int_0^t N(s)ds|N(t)\right) + \mathbb{V}ar\left(\mathbb{E}\int_0^t N(s)ds|N(t)\right) = \frac{\lambda t^3}{3}. \tag{1.91}$$

1.8 Iterated Poisson: composition of two independent Poisson processes

We now study the composition of two independent Poisson processes N_α and N_β of rates λ_α and λ_β (for details see [13]). The process (iterated Poisson)

$$\mathcal{M}(t) \equiv N_\alpha\left(N_\beta(t)\right) \tag{1.92}$$

is a Poisson process sampled at the random time $N_\beta(t)$. Thus $\mathcal{M}(t)$ can be regarded as a time changed Poisson process and is a process with jumps of integer-valued random size.
The following relationship between the iterated Poisson $\mathcal{M}(t)$ and a special compound Poisson process (compound Poisson-Poisson) holds:

$$N_\alpha\left(N_\beta(t)\right) \overset{D}{\sim} \sum_{j=1}^{N_\beta(t)} N_j \tag{1.93}$$

where the N_j are i.i.d. Poisson distributed random variables with parameter λ_α.
Theorem 1.6: The iterated process $\mathcal{M}(t)$ has distribution:

$$Pr\left\{\mathcal{M}(t) = k\right\} = \frac{\lambda_\alpha^k}{k!} e^{-\lambda_\beta t(1-e^{-\lambda_\alpha})} \underbrace{B_k\left(e^{-\lambda_\alpha}\lambda_\beta t\right)}_{\text{Bell polynomials}} \tag{1.94}$$

where the Bell polynomials[4] are defined as $B_k(x) = e^{-x}\sum\limits_{r=0}^{+\infty}\frac{r^k x^r}{r!}$.

Proof. In fact

$$\Pr\{\mathcal{M}(t) = k\} = \mathbb{E}[\Pr\{N_\alpha(N_\beta(t)) = k \mid N_\beta(t)\}] = \sum_{h=0}^{\infty}\underbrace{\Pr\{N_\alpha(h) = k\}\Pr\{N_\beta(t) = h\}}_{\text{joint distribution}}$$

$$= \sum_{h=0}^{\infty} e^{-\lambda_\alpha h}\frac{(\lambda_\alpha h)^k}{k!} e^{-\lambda_\beta t}\frac{(\lambda_\beta t)^h}{h!} = e^{-\lambda_\beta t}\frac{\lambda_\alpha^k}{k!}\sum_{h=0}^{\infty} e^{-\lambda_\alpha h} h^k \frac{(\lambda_\beta t)^h}{h!} = e^{-\lambda_\beta t}\frac{\lambda_\alpha^k}{k!}\sum_{h=0}^{\infty} h^k\frac{(e^{-\lambda_\alpha}\lambda_\beta t)^h}{h!}$$

$$= e^{-\lambda_\beta t}\frac{\lambda_\alpha^k}{k!} e^{e^{-\lambda_\alpha}\lambda_\beta t} \underbrace{e^{-e^{-\lambda_\alpha}\lambda_\beta t}\sum_{h=0}^{\infty} h^k \frac{(e^{-\lambda_\alpha}\lambda_\beta t)^h}{h!}}_{\text{Bell polynomials } \mathcal{B}_k(e^{-\lambda_\alpha}\lambda_\beta t)} = \frac{\lambda_\alpha^k}{k!} e^{-\lambda_\beta t(1-e^{-\lambda_\alpha})} B_k(e^{-\lambda_\alpha}\lambda_\beta t) \tag{1.95}$$

[4]They are also known as Touchard polynomials or exponential Bell polynomials. They represent the moments of the Poisson distribution, i.e. $B_k(\lambda) = \sum_{n=0}^{\infty} n^k \cdot e^{-\lambda}\frac{\lambda^n}{n!}$, where λ is the intensity of the Poisson r.v. In R one can refer to the function *eBellPol* of the package *kStatistics*.

Note further that this iterated Poisson process can count the total number of claims arising from a Poisson number of cars N_β having *each* a Poisson number of accidents N_j. It can be also noted that we can define a compound Poisson-Binomial model, $\sum_{j=1}^{N_\beta(t)} B_j(p)$ counting the total number of fatal crashes arising from a Poisson number of accidents N_β, where a Bernoulli variable $B_j(p)$ takes value 1 if the accident is fatal and zero if not.

Theorem 1.7: The mean and variance of $\mathcal{M}(t)$ are

$$\begin{aligned} \mathbb{E}[\mathcal{M}(t)] &= \lambda_\alpha \lambda_\beta t \\ \mathbb{V}\text{ar}[\mathcal{M}(t)] &= \lambda_\alpha(\lambda_\alpha + 1)\lambda_\beta t > \mathbb{E}[\mathcal{M}(t)] \end{aligned} \tag{1.96}$$

Proof. From the definition

$$\begin{aligned} \mathbb{E}[\mathcal{M}(t)] = \mathbb{E}[N_\alpha(N_\beta(t))] = \mathbb{E}_{N_\beta}[\mathbb{E}[N_\alpha(N_\beta(t)) \mid N_\beta(t)]] = \mathbb{E}[\lambda_\alpha N_\beta(t)] = \lambda_\alpha \lambda_\beta t \\ = \mathbb{V}ar[\mathcal{M}(t)] = \mathbb{E}[\mathcal{M}^2(t)] - \mathbb{E}^2[\mathcal{M}(t)] \end{aligned} \tag{1.97}$$

and

$$\begin{aligned} \mathbb{E}[\mathcal{M}^2(t)] = \mathbb{E}\left[N_\alpha^2(N_\beta(t))\right] \\ = \mathbb{E}_{N_\beta}[\mathbb{E}[N_\alpha^2(N_\beta(t)) \mid N_\beta(t)]] \\ = \sum_{h=0}^{\infty} \mathbb{E}\left[N_\alpha^2(h)\right] \Pr\{N_\beta(t) = h\} = \sum_{h=0}^{\infty} \left[\lambda_\alpha^2 h^2 + \lambda_\alpha h\right] e^{-\lambda_\beta t} \frac{(\lambda_\beta t)^h}{h!} \\ = \lambda_\alpha^2 \sum_{h=0}^{\infty} h^2 e^{-\lambda_\beta t} \frac{(\lambda_\beta t)^h}{h!} + \lambda_\alpha \sum_{h=0}^{\infty} h e^{-\lambda_\beta t} \frac{(\lambda_\beta t)^h}{h!} \\ = \lambda_\alpha^2 \left[\lambda_\beta^2 t^2 + \lambda_\beta t\right] + \lambda_\alpha \lambda_\beta t \end{aligned} \tag{1.98}$$

and

$$\begin{aligned} \mathbb{V}ar[\mathcal{M}(t)] = \lambda_\alpha^2 \lambda_\beta^2 t^2 + \lambda_\alpha^2 \lambda_\beta t + \lambda_\alpha \lambda_\beta t - \lambda_\alpha^2 \lambda_\beta^2 t^2 \\ = \lambda_\alpha \lambda_\beta t(\lambda_\alpha + 1) \end{aligned} \tag{1.99}$$

□

It is interesting to note that the iterated Poisson process is overdispersed having the variance greater than the mean.

For the iterated Poisson process the distribution of the first passage time through a level k, which is defined as

$$T_k^{(\alpha,\beta)} = \inf\{s > 0 : \mathcal{M}(t) = N_\alpha(N_\beta(t)) = k\}, \quad k \geq 1 \tag{1.100}$$

is of great interest. In [10] the authors derive the distribution of the hitting time T_k as follows:

$$\begin{aligned} \Pr\{T_k^{(\alpha,\beta)} \in ds\} \\ = e^{-\lambda_\alpha} \frac{\lambda_\alpha^k}{k!} \lambda_\beta e^{-\lambda_\beta s} ds \sum_{j=0}^{\infty} e^{-\lambda_\alpha j}[(j+1)^k - j^k] \frac{(\lambda_\beta s)^j}{j!}, \quad s > 0. \end{aligned} \tag{1.101}$$

The distribution of T_k reads (cf. Fig. 1.7)

$$Pr\left(T_k \in ds\right) \;=\; ds \lambda_\beta e^{-\lambda_\alpha} e^{-\lambda_\beta s} \tfrac{\lambda_\alpha^k}{k!} \textstyle\sum_{j=0}^{\infty} e^{-\lambda_\alpha j} \left[(j+1)^k - j^k\right] \tfrac{(\lambda_\beta s)^j}{j!} \quad s > 0. \tag{1.102}$$

For the hitting probabilities $Pr(T_k < \infty)$ we have that

$$Pr(T_k < \infty) \quad = \quad e^{-\lambda_\alpha} \frac{\lambda_\alpha^k}{k!} \sum_{j=0}^{\infty} e^{-\lambda_\alpha j} \left[(j+1)^k - j^k \right] < 1 \quad \forall k \tag{1.103}$$

and, in particular,

$$Pr(T_1 < \infty) \quad = \quad \frac{\lambda_\alpha e^{-\lambda_\alpha}}{1 - e^{-\lambda_\alpha}} < 1. \tag{1.104}$$

It means that there is a positive probability of never reaching level k for the iterated Poisson process.

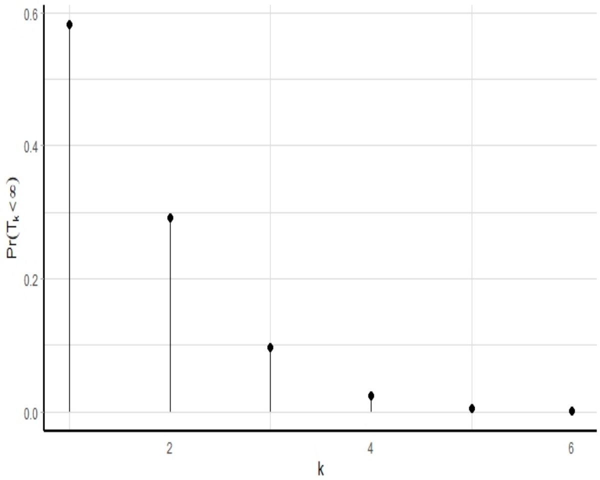


Figure 1.7: Iterated Poisson process: Probability mass function of the hitting time at time $t = 1$ against various levels k of upcrossing and $\lambda_\alpha = \lambda_\beta = 1$.

Chapter 2

Birth processes

2.1 The nonlinear case

In order to describe the population dynamics, an effective model based on the assumption that the probability of new offsprings depends on the current size of the population, is now presented. We assume also, for the sake of simplicity, that at time $t = 0$ there is only one progenitor.
Let $\mathcal{N}(t)$, $t > 0$, be the size population at time t; its probabilistic evolution is governed by the following rules $(k \geq 1)$:

1. $\Pr\{\mathcal{N}(0) = 1\}$=1;
2. $\Pr\{\mathcal{N}(t, t + dt] = 1 \mid \mathcal{N}(t) = k\} = \lambda_k dt + o(dt)$;
3. $\Pr\{\mathcal{N}(t, t + dt] = 0 \mid \mathcal{N}(t) = k\} = 1 - \lambda_k dt + o(dt)$;
4. $\Pr\{\mathcal{N}(t, t + dt] > 1 \mid \mathcal{N}(t) = k\} = o(dt)$,

where $\lambda_k > 0$ are the time-independent birth-rates.

Remark 2.1 A more general birth process is obtained by assuming that the birth rates are time dependent, i.e. $\lambda_k = \lambda_k(t)$.

Remark 2.2 Note that if $\lambda_k = \lambda\ \forall k$ the increments are independent and the process is a homogeneous Poisson process. Note also that this birth process cannot model the population dynamics of male/female type, because here there is essentially a single "sex" generating by parthenogenesis.

Note that the independence of increments, in this case, is no longer holding (cf. Fig. 2.1).
The state probabilities are:

$$p_k(t) = Pr\left\{\mathcal{N}(t) = k \mid \mathcal{N}(0) = 1\right\}, \tag{2.1}$$

satisfy the difference-differential equation

$$\frac{d}{dt}p_k(t) = -\lambda_k p_k(t) + \lambda_{k-1}p_{k-1}(t) \qquad t > 0 \tag{2.2}$$

with initial conditions

$$p_k(0) = \begin{cases} 1 & k = 1 \\ 0 & k \geq 2 \end{cases} \tag{2.3}$$

The solution of (2.2)-(2.3) is carried out by means of a recursive procedure.
If $k = 1$, the solution of

$$\begin{cases} \frac{d}{dt} p_1(t) = -\lambda_1 p_1(t), \\ p_1(0) = 1, \end{cases} \tag{2.4}$$

is $p_1(t) = e^{-\lambda_1 t}$.
To obtain $p_2(t)$, it is necessary to solve the non-homogeneous linear equation

$$\begin{cases} \frac{d}{dt} p_2(t) = -\lambda_2 p_2(t) + \lambda_1 e^{-\lambda_1 t}, \\ p_2(0) = 0, \end{cases} \tag{2.5}$$

that provides

$$p_2(t) = e^{-\lambda_2 t} \left\{ \int_0^t \lambda_1 e^{-\lambda_1 s} e^{\lambda_2 s} \right\} = \lambda_1 e^{-\lambda_2 t} \left[\frac{e^{(\lambda_2 - \lambda_1)s}}{\lambda_2 - \lambda_1} \Bigg|_{s=0}^{s=t} \right] = \lambda_1 \left[\frac{e^{-\lambda_1 t}}{\lambda_2 - \lambda_1} + \frac{e^{-\lambda_2 t}}{\lambda_1 - \lambda_2} \right]. \tag{2.6}$$

In order to understand how the distribution $p_k(t)$ emerges, we consider the case $k = 3$ that is

$$\begin{cases} \frac{d}{dt} p_3(t) = -\lambda_3 p_3(t) + \lambda_2 \lambda_1 \left[\frac{e^{-\lambda_1 t}}{\lambda_2 - \lambda_1} + \frac{e^{-\lambda_2 t}}{\lambda_1 - \lambda_2} \right], \\ p_3(0) = 0. \end{cases} \tag{2.7}$$

By proceeding as for (2.5) we have that

$$\begin{aligned} p_3(t) &= e^{-\lambda_3 t} \left[\int_0^t \lambda_2 \lambda_1 \left(\frac{e^{-\lambda_1 s}}{\lambda_2 - \lambda_1} + \frac{e^{-\lambda_2 s}}{\lambda_1 - \lambda_2} \right) e^{\lambda_3 s} ds \right] \\ &= e^{-\lambda_3 t} \lambda_1 \lambda_2 \left[\frac{e^{(\lambda_3 - \lambda_1)t} - 1}{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)} + \frac{e^{(\lambda_3 - \lambda_2)t} - 1}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)} \right] \\ &= \lambda_1 \lambda_2 \left[\frac{e^{-\lambda_1 t}}{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)} + \frac{e^{-\lambda_2 t}}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)} - \frac{e^{-\lambda_3 t}}{\lambda_2 - \lambda_1} \left(\frac{1}{\lambda_3 - \lambda_1} - \frac{1}{\lambda_3 - \lambda_2} \right) \right] \\ &= \lambda_1 \lambda_2 \left[\frac{e^{-\lambda_1 t}}{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)} + \frac{e^{-\lambda_2 t}}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)} + \frac{e^{-\lambda_3 t}}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \right] \\ &= \prod_{j=1}^{2} \lambda_j \left[\sum_{m=1}^{3} \frac{e^{-\lambda_m t}}{\prod_{l=1, l \neq m}^{3} (\lambda_l - \lambda_m)} \right] \end{aligned} \tag{2.8}$$

Formula (2.8) suggests that the general expression for the probabilities $p_k(t)$ has the form

$$p_k(t) = \begin{cases} \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{e^{-\lambda_m t}}{\prod_{l=1, l \neq m}^{k} (\lambda_l - \lambda_m)}, & k > 1, \\ e^{-\lambda_1 t}, & k = 1. \end{cases} \tag{2.9}$$

Proceeding by induction, we must solve the following Cauchy problem

$$\begin{cases} \frac{d}{dt}p_k(t) = -\lambda_k p_k(t) + \lambda_{k-1}\prod_{j=1}^{k-2}\lambda_j \sum_{m=1}^{k-1} \frac{e^{-\lambda_m t}}{\prod_{l=1,l\neq m}^{k}(\lambda_l-\lambda_m)}, \quad k>1, \\ p_k(0)=0. \end{cases} \tag{2.10}$$

The solution to (2.10) is obtained by applying the formula for the solution of first-order linear non-homogeneous equations[1] which yields:

$$\begin{aligned} p_k(t) &= e^{-\lambda_k t}\left[\int_0^t \prod_{j=1}^{k-1}\lambda_j \sum_{m=1}^{k-1} \frac{e^{-\lambda_m s}}{\prod_{l=1,l\neq m}^{k}(\lambda_l-\lambda_m)} e^{\lambda_k s} ds\right] \\ &= e^{-\lambda_k t}\left[\prod_{j=1}^{k-1}\lambda_j \sum_{m=1}^{k-1} \frac{e^{(\lambda_k-\lambda_m)t}-1}{\prod_{l=1,l\neq m}^{k}(\lambda_l-\lambda_m)(\lambda_k-\lambda_m)}\right] \\ &= \prod_{j=1}^{k-1}\lambda_j \left[\sum_{m=1}^{k-1} \frac{e^{-\lambda_m t}}{\prod_{l=1,l\neq m}^{k}(\lambda_l-\lambda_m)} - e^{-\lambda_k t}\sum_{m=1}^{k-1}\frac{1}{\prod_{l=1,l\neq m}^{k}(\lambda_l-\lambda_m)}\right] \\ &= \prod_{j=1}^{k-1}\lambda_j \left[\sum_{m=1}^{k} \frac{e^{-\lambda_m t}}{\prod_{l=1,l\neq m}^{k}(\lambda_l-\lambda_m)}\right], \end{aligned} \tag{2.11}$$

since[2]

$$-\sum_{m=1}^{k-1}\frac{1}{\prod_{l=1,l\neq m}^{k}(\lambda_l-\lambda_m)} = \frac{1}{\prod_{l=1,l\neq k}^{k}(\lambda_l-\lambda_k)} \tag{2.12}$$

In order to retrieve $p_3(t)$ of formula (2.8) set $k=3$ in (2.9).
Equation (2.2) can also be solved by applying the Laplace transform

$$\int_0^\infty e^{-\mu t}p_k(t)dt = \{\mathcal{L}p_k\}(\mu), \tag{2.13}$$

which provides the relationship (from (2.10))

$$\{\mathcal{L}p_k\}(\mu) = \frac{\lambda_{k-1}}{\mu+\lambda_k}\{\mathcal{L}p_{k-1}\}(\mu) = \prod_{j=1}^{k-1}\lambda_j \prod_{j=1}^{k}\frac{1}{\mu+\lambda_j}. \tag{2.14}$$

By evaluating the Laplace transform of (2.9) and by comparing the result obtained with (2.14) we have that

$$\prod_{j=1}^{k}\frac{1}{\mu+\lambda_j} = \sum_{m=1}^{k}\frac{1}{\prod_{l=1,l\neq m}^{k}(\lambda_l-\lambda_m)}\frac{1}{\mu+\lambda_m}. \tag{2.15}$$

Remark 2.3 If $\sum_{m=1}\frac{1}{\lambda_m}=\infty$ then $\Pr\{\mathcal{N}(t)<\infty\}=1$, so that the size population cannot explode in a finite time. This means that the birth rates must increase slowly so that the sequence $1/\lambda_m$, $m\geq 1$, forms a divergent series.

[1] It is trivial to note that the terms correspond to the form $y'(t)+a(t)=b(t)$, where, for all $k>1$, $a(t)\equiv\lambda_k$ and $b(t)\equiv\lambda_{k-1}\prod_{j=1}^{k-2}\lambda_j\sum_{m=1}^{k-1}\frac{e^{-\lambda_m}}{\prod_{l=1,l\neq m}^{k}(\lambda_l-\lambda_m)}$.

[2] The result obtains after application of the Laplace expansion for determinants.

Theorem 2.1: The time between the k-th and $(k+1)$-th birth has distribution

$$\Pr\{T_k \in ds\} = \lambda_k e^{-\lambda_k s} ds, \qquad s > 0. \tag{2.16}$$

Let $Z_k = T_1 + \cdots + T_k$ the waiting time for k-th birth. We have the following relation:

$$\Pr\{T_1 + \cdots + T_k \in dt\} = \int_0^t \Pr\{T_1 + \cdots + T_{k-1} \in ds\}\Pr\{T_k \in d(t-s)\}, \tag{2.17}$$

where

$$\Pr\{T_1 + \cdots + T_{k-1} \in ds\}/ds = \frac{d}{ds}\Pr\{\mathcal{N}(s) \geq k\}. \tag{2.18}$$

Consider that $\mathcal{N}$ includes also the progenitor[3].
By applying the Laplace transform to both members of (2.17), we have that

$$\begin{aligned}
&\int_0^\infty e^{-\mu t}\Pr\{T_1 + \cdots + T_k \in dt\} \\
&= \int_0^\infty e^{-\mu t} dt \int_0^t \Pr\{T_k \in d(t-s)\}\Pr\{T_1 + \cdots + T_{k-1} \in ds\} \\
&= \int_0^\infty \Pr\{T_1 + \cdots + T_{k-1} \in ds\} \int_s^\infty e^{-\mu t}\Pr\{T_k \in d(t-s)\} \\
&= \int_0^\infty e^{-\mu s}\Pr\{T_1 + \cdots + T_{k-1} \in ds\} \int_0^\infty e^{-\mu y}\Pr\{T_k \in dy\} \\
&= \prod_{j=1}^k \int_0^\infty e^{-\mu s}\Pr\{T_j \in ds\} = \prod_{j=1}^k \frac{\lambda_j}{\mu + \lambda_j}.
\end{aligned} \tag{2.19}$$

Each factor in (2.19) is the Laplace transform of (2.11).
Furthermore

$$\Pr\{T_1 < s\} = \Pr\{N(s) \geq 2\} \tag{2.20}$$

and taking the derivative

$$\frac{\Pr\{T_1 \in ds\}}{ds} = \frac{d}{ds}\Pr\{N(s) \geq 2\} = \frac{d}{ds}(1 - \Pr\{N(s) = 1\}) = \frac{d}{ds}(1 - e^{-\lambda_1 s}) = \lambda_1 e^{-\lambda_1 s} \tag{2.21}$$

and

$$\Pr\{T_1 + T_2 < s\} = \Pr\{N(s) \geq 3\} = 1 - \Pr\{N(s) = 1\} - \Pr\{N(s) = 2\} \tag{2.22}$$

and deriving:

$$\begin{aligned}
\Pr\{T_1 + T_2 \in ds\}/ds &= \frac{d}{ds}\{1 - \Pr\{N(s) = 1\} - \Pr\{N(s) = 2\}\} \\
&= \lambda_1 e^{-\lambda_1 s} + \lambda_1 \left[\lambda_1 \frac{e^{-\lambda_1 s}}{\lambda_2 - \lambda_1} + \lambda_2 \frac{e^{-\lambda_2 s}}{\lambda_1 - \lambda_2}\right] \\
&= \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 s} - e^{-\lambda_2 s}\right).
\end{aligned} \tag{2.23}$$

[3] i.e., k includes one progenitor and $k-1$ offsprings.

Note that (2.19), for $k = 2$, becomes

$$\int_0^\infty e^{-\mu t} \Pr\{T_1 + T_2 \in dt\} = \frac{\lambda_1 \lambda_2}{(\lambda_1 + \mu)(\lambda_2 + \mu)}, \tag{2.24}$$

and corresponds to the Laplace transform of (2.23).

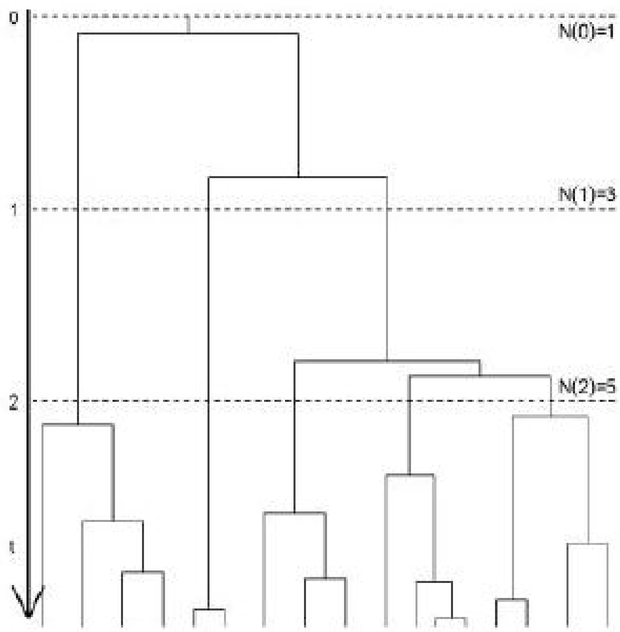


Figure 2.1: A general sample path of a birth processes.

2.2 Linear case (Yule-Furry)

In this section we denote with $\hat{N}(t)$, $t > 0$, the size of the population at time t of a linear birth process (called also Yule–Furry process). In this case the birth rates become $\lambda_k = \lambda k$, con $\lambda > 0$. The distribution (2.9) becomes relatively simple. By observing that

$$\begin{aligned} \prod_{l=1, l\neq m}^{k} (l\lambda - m\lambda) = & \\ & = \lambda^{k-1} \left[(1-m)(2-m)\dots(m-1-m)(m+1-m)\dots(k-m)\right] \\ & = \lambda^{k-1}(-1)^{m-1}(m-1)!(k-m)! \ , \end{aligned} \tag{2.25}$$

formula (2.9) can be rewritten as

$$
\begin{aligned}
p_k(t) &= \lambda^{k-1}(k-1)! \sum_{m=1}^{k} \frac{e^{-\lambda mt}}{\lambda^{k-1}(m-1)!(k-m)!}(-1)^{m-1} \\
&= \sum_{m=1}^{k} e^{-\lambda mt}(-1)^{m-1}\frac{(k-1)!}{(m-1)!(k-m)!} = \sum_{m=0}^{k-1} e^{-\lambda t(m+1)}(-1)^m \frac{(k-1)!}{m!(k-m-1)!} \\
&= e^{-\lambda t}\sum_{m=0}^{k-1}\binom{k-1}{m}(-1)^m e^{-\lambda tm} = e^{-\lambda t}\left(1-e^{-\lambda t}\right)^{k-1}, \qquad k \geq 1,
\end{aligned}
\tag{2.26}
$$

which is a geometric distribution (cf. Fig. 2.2).

Remark 2.4 Note that if the progenitors are $n_0 > 1$ the random number $\hat{N}(t)$ of components of the population becomes a Negative Binomial distribution (the sum of n_0 independent Geometric r.v.'s):

$$p_k(t) = \tbinom{k-1}{k-n_0}e^{-\lambda tn_0}(1-e^{-\lambda t})^{k-n_0}, \qquad k \geq n_0,$$

(cf. Fig. 2.3 and Fig. 2.4).

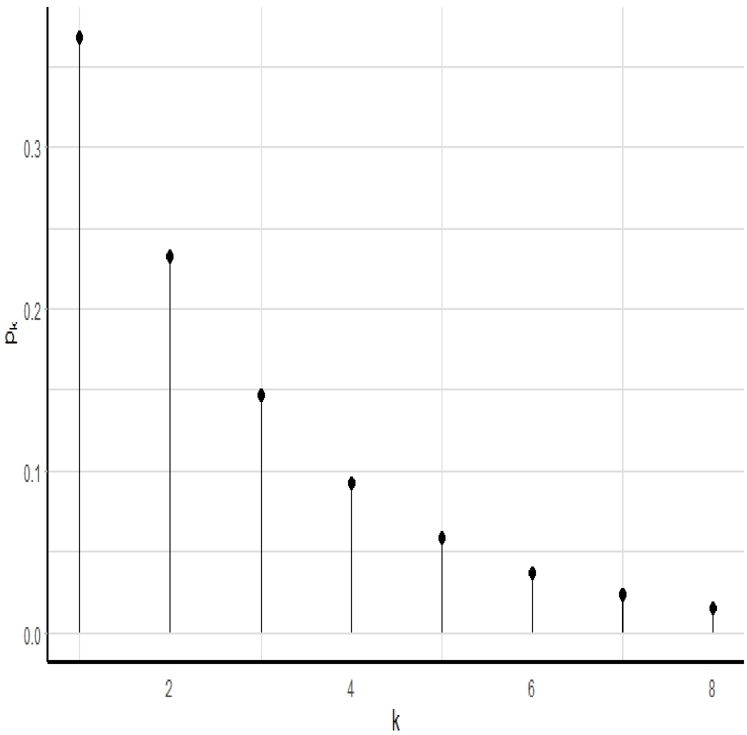


Figure 2.2: State probabilities $\{p_k(t),\ k \geq 1\}$ of a pure birth process (linear case), with one progenitor ($n_0 = 1$) and $\lambda = 1$; $t = 1$.

The probabilities (2.26) can be obtained directly by solving the following equations, recursively

$$\frac{d}{dt}p_k(t) = -\lambda k p_k(t) + \lambda(k-1)p_{k-1}(t) \tag{2.27}$$

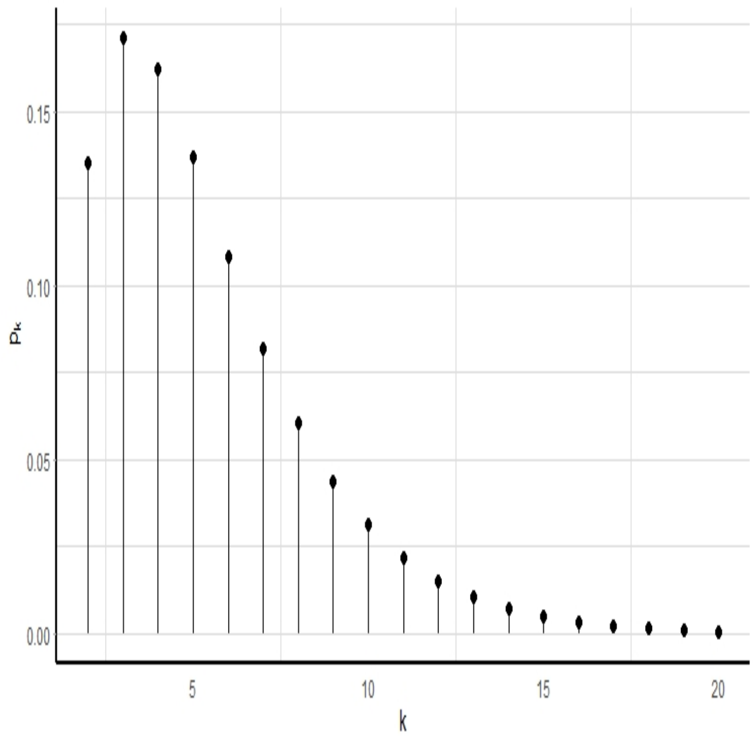


Figure 2.3: State probabilities $\{p_k(t),\ k \geq 1\}$ of a pure birth process (linear case), for $n_0 = 2$ progenitors (Negative Binomial) and $\lambda = 1$; $t = 1$.

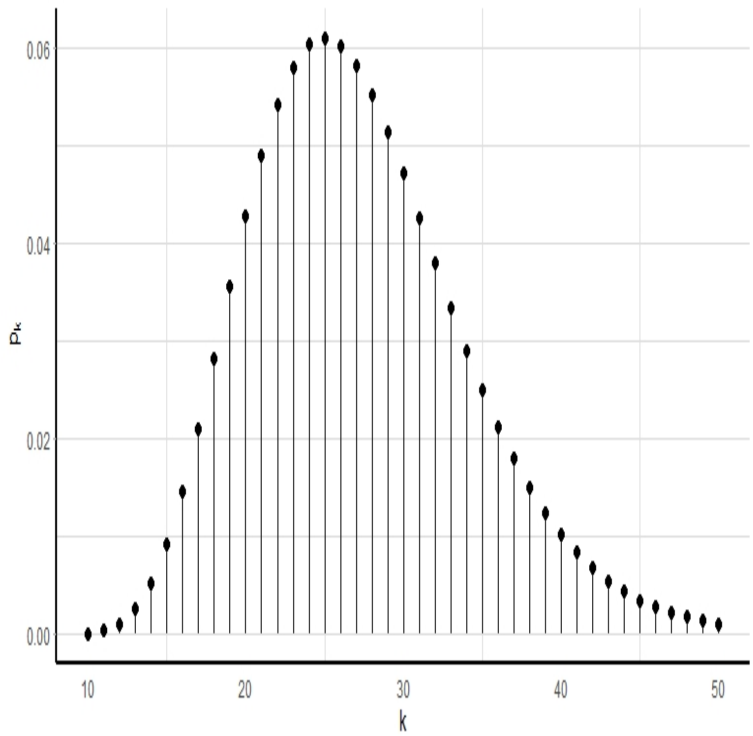


Figure 2.4: State probabilities $\{p_k(t),\ k \geq 1\}$ of a pure birth process (linear case), $n_0 = 10$ progenitors (Negative Binomial) and $\lambda = 1$; $t = 1$.

with initial conditions

$$p_k(0) = \begin{cases} 1, & k = 1, \\ 0, & k \geq 2. \end{cases} \tag{2.28}$$

By using the probability generating function

$$G(u,t) = \sum_{k=1}^{\infty} u^k p_k(t), \quad |u| \le 1 \tag{2.29}$$

we can pass from (2.27) to the following partial differential equation

$$\frac{\partial}{\partial t} G(u,t) = \lambda u(u-1) \frac{\partial}{\partial u} G(u,t) \tag{2.30}$$

with initial condition $G(u,0) = u$.
The general solution of (2.30) is

$$G(u,t) = f\left(e^{-\lambda t} \frac{u}{1-u}\right) \tag{2.31}$$

where $f \in \mathbb{C}^1$ is a differentiable function. The initial condition permits us to obtain the explicit form of f because

$$G(u,0) = u = f\left(\frac{u}{1-u}\right) \tag{2.32}$$

and by setting $\frac{u}{1-u} = v$ we see that $f(v) = \frac{v}{1+v}$. From (2.31) we then have that

$$\begin{aligned} G(u,t) &= \frac{e^{-\lambda t}\frac{u}{1-u}}{1+\frac{u}{1-u}e^{-\lambda t}} = \frac{ue^{-\lambda t}}{1-u\left(1-e^{-\lambda t}\right)} \\ &= ue^{-\lambda t} \sum_{k=0}^{\infty} u^k \left(1-e^{-\lambda t}\right)^k \\ &= \sum_{k=1}^{\infty} u^k e^{-\lambda t} \left(1-e^{-\lambda t}\right)^{k-1} \end{aligned} \tag{2.33}$$

and this confirms (2.26).
From the probability generating function, by means of straightforward calculation, we can obtain that

$$\begin{aligned} \mathbb{E}[\hat{N}(t)] &= e^{\lambda t} \\ \mathbb{V}\mathrm{ar}[\hat{N}(t)] &= e^{\lambda t}\left(e^{\lambda t}-1\right). \end{aligned} \tag{2.34}$$

Thus a linear birth process has mean and variance increasing exponentially and since

$$\sum_{k=1}^{\infty} \frac{1}{\lambda k} = \infty \tag{2.35}$$

$\Pr\{\hat{N}(t) < \infty\} = 1$ for $t > 0$, that is the Yule–Furry process does not explode.

Chapter 3

Death processes

3.1 The nonlinear case

We consider a population with n_0 elements at time $t = 0$. Let $\mathcal{M}(t)$ be the number of elements of the population at time $t > 0$ and we denote by $\mathcal{M}(t, t + dt]$ the r.v. representing the decrease of the population in the time interval $(t,\, t + dt]$.
We suppose that the population can only decrease according to the following probabilistic rules

1. $\Pr\{\mathcal{M}(t) = n_0\} = 1$;
2. $\Pr\{\mathcal{M}(t, t + dt] = -1 \mid \mathcal{M}(t) = k\} = \mu_k dt + o(dt)$, $\mu_k > 0$;
3. $\Pr\{\mathcal{M}(t, t + dt] = 0 \mid \mathcal{M}(t) = k\} = 1 - \mu_k dt + o(dt)$;
4. $\Pr\{\mathcal{M}(t, t + dt] < -1 \mid \mathcal{M}(t) = k\} = o(dt)$,

The last property excludes simultaneous deaths.
The distribution

$$p_k(t) = \Pr\{\mathcal{M}(t) = k \mid \mathcal{M}(0) = n_0\} \tag{3.1}$$

is obtained by the solving the following difference-differential equations

$$\begin{cases} \frac{d}{dt} p_k(t) = -\mu_k p_k(t) + \mu_{k+1} p_{k+1}(t), & 1 \le k < n_0, \\ \frac{d}{dt} p_{n_0}(t) = -\mu_{n_0} p_{n_0}(t), & k = n_0, \\ \frac{d}{dt} p_0(t) = \mu_1 p_1(t), & k = 0. \end{cases} \tag{3.2}$$

The equations (3.2) are obtained by considering that the probability that at time $t + dt$ there are k survivors is the probability that at time t either there are k survivors and in $(t, t + dt]$ no death is recorded or there are $k + 1$ survivors and one of them disappears during the same time interval, i.e.

$$p_k(t + dt) = p_k(t)(1 - \mu_k dt) + p_{k+1}(t)\mu_{k+1} dt + o(dt). \tag{3.3}$$

The second of equations (3.2) comes from

$$p_{n_0}(t + dt) = p_{n_0}(t)(1 - \mu_{n_0} dt) + o(dt) \tag{3.4}$$

and the last of equations (3.2) comes from

$$p_0(t+dt) = p_0(t) + p_1(t)\mu_1 dt + o(dt). \tag{3.5}$$

A relevant feature for the death process is the extinction probability $p_0(t)$. Note that the second and third of equations (3.2) are obtained from the first one for $k = n_0$ and $k = 0$, respectively. The second equation of (3.2) comes from the first one with $\mu_{n_0+1} = 0$ because the population cannot increase in a pure death process.

Theorem 3.1: The probability distribution emerging from (3.2), has the form

$$p_k(t) = \begin{cases} e^{-\mu_{n_0} t}, & k = n_0, \\ \prod_{j=k+1}^{n_0} \mu_j \sum_{m=k}^{n_0} \frac{e^{-\mu_m t}}{\prod_{h=k,h\neq m}^{n_0}(\mu_h - \mu_m)}, & 1 \le k < n_0, \\ 1 - \sum_{m=1}^{n_0} \prod_{h=1,h\neq m}^{n_0} \left(\frac{\mu_h}{\mu_h - \mu_m}\right) e^{-\mu_m t}, & k = 0. \end{cases} \tag{3.6}$$

Proof. The first result is straightforward. As far as the second probability of (3.6) is concerned we restrict ourselves to some specific cases. Equation (3.2), for $k = n_0 - 1$ becomes

$$\begin{aligned} \frac{d}{dt} p_{n_0-1}(t) = & \\ & = -\mu_{n_0-1} p_{n_0-1}(t) + \mu_{n_0} p_{n_0}(t) \\ & = -\mu_{n_0-1} p_{n_0-1}(t) + \mu_{n_0} e^{-\mu_{n_0} t}, \end{aligned} \tag{3.7}$$

and its solution is

$$\begin{aligned} p_{n_0-1}(t) = & \\ & = e^{-t\mu_{n_0-1}} \left\{ \int_0^t \mu_{n_0} e^{-s\mu_{n_0}} e^{s\mu_{n_0-1}} + \text{cost} \right\} \\ & = \mu_{n_0} e^{-t\mu_{n_0-1}} \left\{ \frac{1 - e^{-(\mu_{n_0} - \mu_{n_0-1})t}}{\mu_{n_0} - \mu_{n_0-1}} \right\} \\ & = \mu_{n_0} \left[\frac{e^{-t\mu_{n_0-1}}}{\mu_{n_0} - \mu_{n_0-1}} + \frac{e^{-t\mu_{n_0}}}{\mu_{n_0-1} - \mu_{n_0}} \right], \end{aligned} \tag{3.8}$$

and is equal to (3.6) for $k = n_0 - 1$.
The recursion method permits us to write

$$\frac{d}{dt} p_{n_0-2}(t) = -\mu_{n_0-2} p_{n_0-2}(t) + \mu_{n_0-1}\mu_{n_0} \left[\frac{e^{-t\mu_{n_0-1}}}{\mu_{n_0} - \mu_{n_0-1}} + \frac{e^{-t\mu_{n_0}}}{\mu_{n_0-1} - \mu_{n_0}} \right], \tag{3.9}$$

with initial condition $p_{n_0-2}(0) = 0$.

The solution of (3.9) reads

$$
\begin{aligned}
p_{n_0-2}(t) &= e^{-t\mu_{n_0-2}}\left\{\int_0^t \mu_{n_0-1}\mu_{n_0}\left[\frac{e^{-s\mu_{n_0-1}}}{\mu_{n_0}-\mu_{n_0-1}}+\frac{e^{-s\mu_{n_0}}}{\mu_{n_0-1}-\mu_{n_0}}\right]e^{s\mu_{n_0-2}}ds\right\}\\
&= \mu_{n_0}\mu_{n_0-1}e^{-t\mu_{n_0-2}}\left\{\frac{1-e^{-(\mu_{n_0-1}-\mu_{n_0-2})t}}{(\mu_{n_0}-\mu_{n_0-1})(\mu_{n_0-1}-\mu_{n_0-2})}-\frac{1-e^{-(\mu_{n_0}-\mu_{n_0-2})t}}{(\mu_{n_0}-\mu_{n_0-1})(\mu_{n_0}-\mu_{n_0-2})}\right\}\\
&= \mu_{n_0}\mu_{n_0-1}\left[\frac{e^{-t\mu_{n_0-2}}}{(\mu_{n_0}-\mu_{n_0-1})(\mu_{n_0-1}-\mu_{n_0-2})}-\frac{e^{-t\mu_{n_0-2}}}{(\mu_{n_0}-\mu_{n_0-1})(\mu_{n_0}-\mu_{n_0-2})}\right.\\
&\quad\left.+\frac{e^{-t\mu_{n_0-1}}}{(\mu_{n_0}-\mu_{n_0-1})(\mu_{n_0-2}-\mu_{n_0-1})}+\frac{e^{-t\mu_{n_0}}}{(\mu_{n_0-1}-\mu_{n_0})(\mu_{n_0-2}-\mu_{n_0})}\right]\\
&= \mu_{n_0}\mu_{n_0-1}\left[\frac{e^{-t\mu_{n_0}}}{(\mu_{n_0-1}-\mu_{n_0})(\mu_{n_0-2}-\mu_{n_0})}+\frac{e^{-t\mu_{n_0-1}}}{(\mu_{n_0}-\mu_{n_0-1})(\mu_{n_0-2}-\mu_{n_0-1})}\right.\\
&\quad\left.+\frac{e^{-t\mu_{n_0-2}}}{(\mu_{n_0}-\mu_{n_0-2})(\mu_{n_0-1}-\mu_{n_0-2})}\right].
\end{aligned}
\tag{3.10}
$$

Note that (3.6) for $k = n_0 - 2$ is equal to (3.10). The general case can obtained in a similar way but the details are omitted (see [17]).

3.2 The linear case

The special case of the linear death process is retrieved for $\mu_k = k\mu$, so the mortality rate is proportional to the current size of the population.
By observing that

$$
\begin{aligned}
\prod_{h=k,h\neq m}^{n_0}(\mu_h-\mu_m) &= (\mu_k-\mu_m)\dots(\mu_{m-1}-\mu_m)(\mu_{m+1}-\mu_m)\dots(\mu_{n_0}-\mu_m)\\
&= (k-m)\mu\dots(-1)\mu\cdot 1\cdot\mu\dots(n_0-m)\mu\\
&= (-1)^{m-k}\mu^{m-k}(m-k)!\mu^{n_0-m}(n_0-m)!\\
&= (-1)^{m-k}\mu^{n_0-k}(m-k)!(n_0-m)!
\end{aligned}
\tag{3.11}
$$

and that

$$
\prod_{h=k+1}^{n_0}\mu_h = \mu^{n_0-k}n_0(n_0-1)\dots(n_0-k-1)\frac{h!}{k!} = \mu^{n_0-k}\frac{n_0!}{k!}, \tag{3.12}
$$

formula (3.6) entails that

$$
\begin{aligned}
p_k(t) &= \frac{n_0!}{k!}\sum_{m=k}^{n_0}\frac{e^{-\mu m t}(-1)^{m-k}}{(m-k)!(n_0-m)!}\\
&= \binom{n_0}{k}\sum_{m=k}^{n_0}\binom{n_0-k}{m-k}(-1)^{m-k}e^{-\mu m t}\\
&= \binom{n_0}{k}\sum_{r=0}^{n_0-k}\binom{n_0-k}{r}(-1)^r e^{-\mu(k+r)t}\\
&= \binom{n_0}{k}e^{-\mu k t}\left(1-e^{-\mu t}\right)^{n_0-k}, \quad 0\le k\le n_0
\end{aligned}
\tag{3.13}
$$

so the distribution of the linear death process is Binomial (cf. Fig. 3.1) with parameters n_0 and $e^{-\mu t}$ (see [17]).

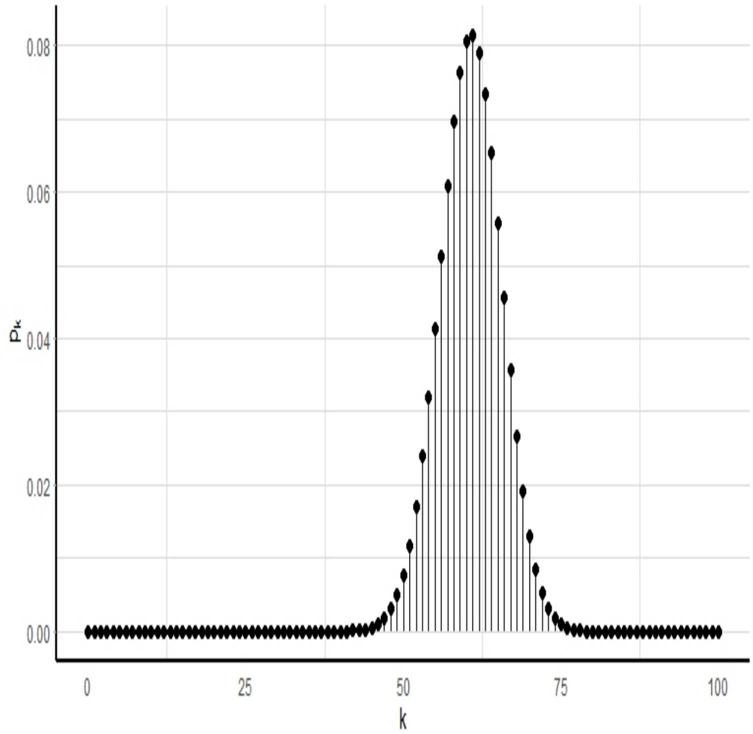


Figure 3.1: State probabilities $\{p_k(t),\, k \geq 1\}$ of a pure-death process (linear case) with initial population size $n_0 = 100$, after $t = 1$ time unit ($\mu = 0.1$). Binomial$(n_0, e^{-\mu t})$.

3.3 The sublinear case

As shown in [17], in this case the infinitesimal death probabilities have the form

$$\Pr\{\mathcal{M}(t, t + dt) = -1 \mid \mathcal{M}(t) = k\} = \mu(n_0 + 1 - k)dt + o(dt) \tag{3.14}$$

where the death rate does not depend on the the number k of survivors, but on the number $n_0-(k-1)$ of deaths occurred in the time interval $(0, t)$, so that for $k = n_0$ the probability (3.14) reduces to is $\mu dt + o(dt)$.
The difference-differential equations governing the survival probabilities $p_k(t)$ are

$$\frac{d}{dt}p_k(t) = -\mu(n_0 - k + 1)p_k(t) + \mu(n_0 - k)p_{k+1}(t) \tag{3.15}$$

subject to

$$\begin{cases} \frac{d}{dt}p_{n_0}(t) = -\mu p_{n_0}(t) \\ \frac{d}{dt}p_0(t) = \mu n_0 p_1(t) \end{cases} \tag{3.16}$$

and

$$p_k(0) = \begin{cases} 1 & k = n_0 \\ 0 & 0 \leq k < n_0 \end{cases} \tag{3.17}$$

with solution given in [17]

$$p_k(t) = \sum_{h=0}^{n_0-k} \binom{n_0 - k}{h} (-1)^h e^{-(h+1)\mu t} \quad 1 \leq k \leq n_0 \tag{3.18}$$

and extinction probability

$$p_0(t) = \sum_{h=0}^{n_0} \binom{n_0}{h} (-1)^h e^{-h\mu t}. \tag{3.19}$$

Chapter 4

The birth-death process

4.1 The nonlinear case

We consider a process where birth and death are admitted so that the population $\mathcal{N}(t)$ can increase and decrease as time passes. We assume that

$$\begin{cases} \Pr\{(\mathcal{N}(t,t+dt)=1 \mid \mathcal{N}(t)=k\} = \lambda_k dt + o(dt) \\ \Pr\{(\mathcal{N}(t,t+dt)=-1 \mid \mathcal{N}(t)=k\} = \mu_k dt + o(dt) \\ \Pr\{(\mathcal{N}(t,t+dt)=0 \mid \mathcal{N}(t)=k\} = 1-(\lambda_k+\mu_k)dt + o(dt) \end{cases} \tag{4.1}$$

where μ_k and λ_k are the death and birth rates.
Furthermore we assume also that

$$\Pr\{\mathcal{N}(t,t+dt)=\pm k \mid \mathcal{N}(t)=k\} = o(dt), k \geq 2 \tag{4.2}$$

i.e., we suppose that clusters of deaths and births occur with a negligible probability. The state probabilities

$$p_k(t) = \Pr\{\mathcal{N}(t)=k \mid \mathcal{N}(0)=1\}, k \geq 0 \tag{4.3}$$

solve the system of equations

$$\frac{dp_k(t)}{dt} = -(\lambda_k+\mu_k)p_k(t) + \lambda_{k-1}p_{k-1}(t) + \mu_{k+1}p_{k+1}(t), \qquad k \geq 0 \tag{4.4}$$

under initial conditions

$$p_k(0) = \begin{cases} 1 & k=1 \qquad \text{(one initial progenitor)} \\ 0 & k \neq 1. \end{cases} \tag{4.5}$$

Equation (4.4) can be obtained in the following manner:

$$\begin{aligned} p_k(t+dt) = & \\ & = p_k(t)(1-\lambda_k dt)(1-\mu_k dt) + p_{k-1}(t)\lambda_{k-1}dt(1-\mu_{k-1}dt) \\ & + p_{k+1}(t)(1-\lambda_{k+1}dt)\mu_{k+1}dt + p_k(t)\lambda_k dt\mu_k dt \\ & = p_k(t) - \lambda_k p_k(t)dt - \mu_k p_k(t)dt + p_{k-1}(t)\lambda_{k-1}dt + p_{k+1}(t)\mu_{k+1}dt \end{aligned} \tag{4.6}$$

By expanding $p_k(t+dt)$, dividing both members by dt we get (4.4).

Remark 4.1. If the Poisson rates are constant, $\lambda_k = \lambda$, $\mu_k = \mu$, the birth-death process becomes the immigration-emigration process.

4.2 The linear case

In the linear case we obtain:

$$\frac{dp_k(t)}{dt} = -(\lambda+\mu)kp_k(t) + \lambda(k-1)p_{k-1}(t) + \mu(k+1)p_{k+1}(t), \tag{4.7}$$

The exact distribution (see [12][1]) $p_k(t) = \Pr\{\mathcal{N}(t) = k\}$, $k \geq 1$ is (cf. Fig. 4.1)

$$p_k(t) = (\lambda-\mu)^2 e^{-(\lambda-\mu)t} \frac{[\lambda(1-e^{-(\lambda-\mu)t})]^{k-1}}{(\lambda-\mu e^{-(\lambda-\mu)t})^{k+1}}, k \geq 1, \mu \neq \lambda, \tag{4.8}$$

which for $\lambda = \mu$ simplifies into (cf. Fig. 4.3)

$$p_k(t) = \frac{(\lambda t)^{k-1}}{(1+\lambda t)^{k+1}}, k \geq 1 \tag{4.9}$$

The extinction probabilities are

$$p_0(t) = \begin{cases} \frac{\mu t}{1+\mu t} & \lambda = \mu \\ \frac{\mu - \mu e^{-(\lambda-\mu)t}}{\lambda - \mu e^{-(\lambda-\mu)t}} & \lambda \neq \mu \end{cases} \tag{4.10}$$

The probability generating function $G(u,t) = \mathbb{E}[u^{\mathcal{N}(t)}]$ of $\mathcal{N}(t)$ satisfies in the linear case the partial differential equation

$$\frac{\partial G(u,t)}{\partial t} = \left[\lambda u^2 - (\lambda+\mu)u + \mu\right] \frac{\partial G(u,t)}{\partial u} = (\lambda u - \mu)(u-1)\frac{\partial G(u,t)}{\partial u}, \qquad |u| \leq 1, t > 0 \tag{4.11}$$

with the initial condition $G(u,0) = u$ (with a single progenitor at time $t = 0$).
The general form of the solution of equation (4.11) is

$$G(u,t) = f\left(e^{-(\lambda-\mu)t}\frac{\lambda u - \mu}{1-u}\right) \tag{4.12}$$

and this can be checked by substituting equation (4.12) into equation (4.11) and assuming that $f \in \mathbb{C}^1$. The argument of equation (4.12) is obtained by the method of auxiliary functions. By

[1] Cf. also Bailey (1964) [1].

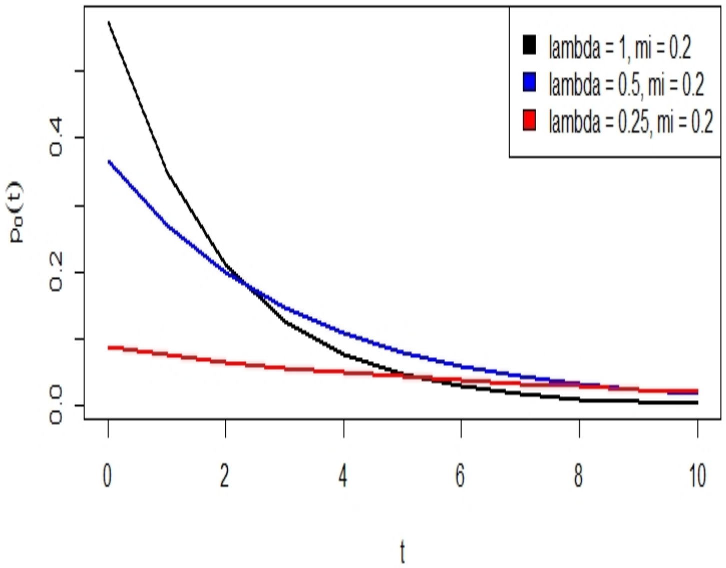


Figure 4.1: Extinction probabilities $p_0(t)$ for (a) $\lambda = 1, \mu = 0.2$, (b) $\lambda = 0.5, \mu = 0.2$, (c) $\lambda = 0.25, \mu = 0.2$.

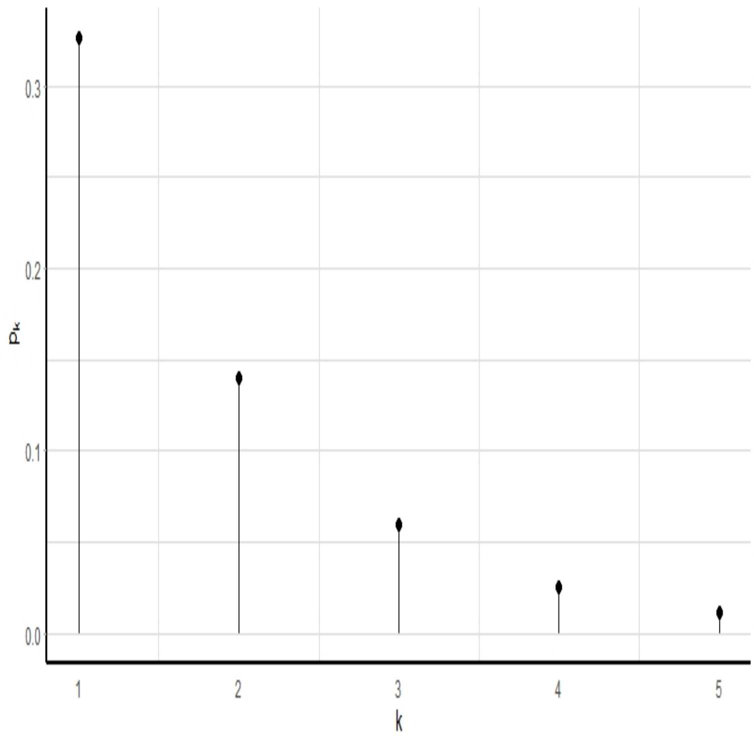


Figure 4.2: Distribution $\{p_k(t),\ k \geq 1\}$, for $\lambda = \mu = 0.75$; $t = 1$.

imposing the initial condition

$$G(u, 0) = \sum_{k=0}^{\infty} u^k p_k(0) = u = f\left(\frac{\lambda u - \mu}{1 - u}\right) \tag{4.13}$$

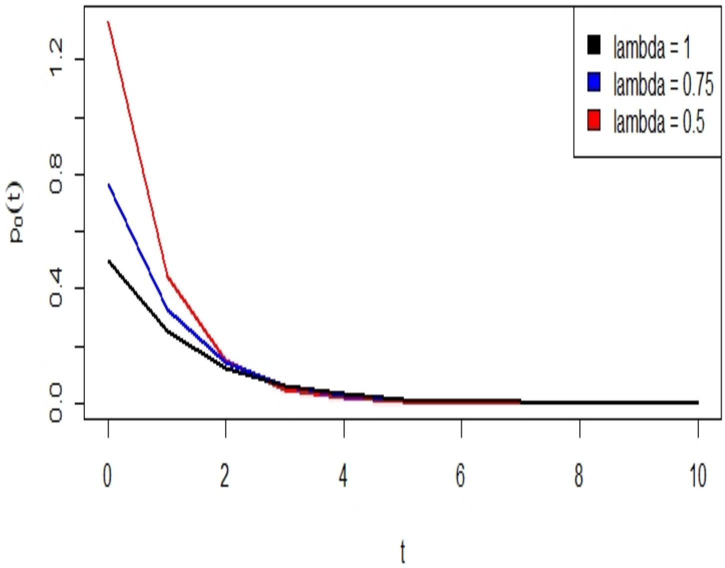


Figure 4.3: Extinction probabilities $p_0(t)$, for $\lambda = 1$, $\lambda = 0.75$ and $\lambda = 0.5$.

we obtain that

$$f(u) = \frac{\mu + u}{\lambda + u} \tag{4.14}$$

because if we set $\frac{\lambda u - \mu}{1-u} = v$, we readily arrive at (4.14).
The p.g.f. for $\lambda \neq \mu$ is therefore

$$G(u,t) = \begin{cases} \frac{\mu + \frac{\lambda u - \mu}{1-u} e^{-(\lambda-\mu)t}}{\lambda + \frac{\lambda u - \mu}{1-u} e^{-(\lambda-\mu)t}} = \frac{\mu(1-u)+(\lambda u-\mu)e^{-(\lambda-\mu)t}}{\lambda(1-u)+(\lambda u-\mu)e^{-(\lambda-\mu)t}} & u > 0 \\ p_0(t) = \frac{\mu - \mu e^{-(\lambda-\mu)t}}{\lambda - \mu e^{-(\lambda-\mu)t}}, & u = 0 \end{cases} \tag{4.15}$$

The limit of the extinction probability (4.15) for $\lambda \to \mu$ reads (cf. Fig. 4.2)

$$\lim_{\lambda \to \mu} \frac{\mu - \mu e^{-(\lambda-\mu)t}}{\lambda - \mu e^{-(\lambda-\mu)t}} = \frac{\mu t}{1 + \mu t}. \tag{4.16}$$

4.3 The extinction probability and the Riccati equation

The equation governing the extinction probability is

$$p_0(t) = \int_0^t e^{-\lambda s} \mu e^{-\mu s} ds + \int_0^t \lambda e^{-\lambda s} e^{-\mu s} p_0^2(t-s) ds. \tag{4.17}$$

Equation (4.17) is obtained by considering that

$$\int_0^t e^{-\lambda s} \mu e^{-\mu s} ds \tag{4.18}$$

represents the probability that in $(0, s)$ the initial particle will survive up to time s with probability

$e^{-\mu s}$(and no offspring was born, with probability $e^{-\lambda s}$), and that it will die during $[s, s+ds)$ with probability μds.
The second term of (4.17) considers that the initial progenitor with probability $e^{-\mu s}$ survives up to time s (and no birth is registered in $(0,s)$ with probability $e^{-\lambda s}$), an additional offspring is generated in $[s, s+ds)$ with probability λds and both components of the population will die during the remaining time $[s,t]$, independently, with probability $p_0^2(t-s)$.
Hence, by deriving with respect to t, we obtain

$$p_0'(t) = \mu e^{-(\lambda+\mu)t} + \lambda e^{-(\lambda+\mu)t} p_0^2(0) + \lambda \int_0^t e^{-(\lambda+\mu)s} \frac{d}{dt} p_0^2(t-s) ds \tag{4.19}$$

Equation (4.19) is then simplified as follows:

$$\begin{aligned} p_0'(t) &= \mu e^{-(\lambda+\mu)t} + \lambda e^{-(\lambda+\mu)t} p_0^2(0) - \lambda \int_0^t e^{-(\lambda+\mu)s} \frac{d}{ds} p_0^2(t-s) ds \\ &= \mu e^{-(\lambda+\mu)t} - \lambda e^{-(\lambda+\mu)s} p_0^2(t-s)|_{s=0}^{s=t} - \lambda(\lambda+\mu) \int_0^t e^{-(\lambda+\mu)s} p_0^2(t-s) ds \\ &= \mu e^{-(\lambda+\mu)t} + \lambda p_0^2(t) - (\lambda+\mu) \left[p_0(t) - \mu \int_0^t e^{-(\lambda+\mu)s} ds \right] \\ &= -(\lambda+\mu) p_0(t) + \lambda p_0^2(t) + \mu e^{-(\lambda+\mu)t} + [-\mu e^{-(\lambda+\mu)s}]_{s=0}^{s=t} \\ &= \mu - (\lambda+\mu) p_0(t) + \lambda p_0^2(t). \end{aligned} \tag{4.20}$$

Hence the extinction probability $p_0(t)$, $t > 0$, satisfies the non-homogeneous Riccati equation

$$\frac{dp_0(t)}{dt} + (\lambda+\mu) p_0(t) = \lambda p_0^2(t) + \mu \tag{4.21}$$

An alternative derivation is the following.
By considering the process in the initial interval $[0, ds)$ and writing

$$p_0(t) = \mu ds + (1-\mu ds)(1-\lambda ds) p_0(t-ds) + (1-\mu ds) \lambda ds p_0^2(t-ds), \tag{4.22}$$

we are supposing that

- 1. the process becomes extincted in the initial interval $[0, ds)$ with probability μds;
- 2. the process survives and does not generate an additional offspring in the initial interval ds, and gets extincted in the interval $[ds, t)$ with probability

$$(1-\mu ds)(1-\lambda ds) p_0(t-ds); \tag{4.23}$$

- 3. the process generates an additional component with probability λds and in the remaining time $[ds, t)$ the two components (independently) disappear .

Neglecting infinitesimal of higher order we have that

$$\frac{p_0(t) - p_0(t-ds)}{ds} = \mu - (\lambda+\mu) p_0(t-ds) + \lambda p_0^2(t-ds) \tag{4.24}$$

and for $ds \to 0$ the Riccati equation emerges and is satisfied by the extinction probability (4.10) above.
Functions (4.15) and (4.16) are solutions of the Riccati equation of the next section.
For $\lambda = \mu$ we have that

$$\frac{d}{dt}p_0(t) + 2\mu p_0(t) = \mu p_0^2(t) + \mu \tag{4.25}$$

is satisfied by (4.16).
Finally we observe that

$$\lim_{t\to\infty} p_0(t) = \begin{cases} \frac{\mu}{\lambda}, & \lambda > \mu \\ 1, & \lambda = \mu \\ 1, & \lambda < \mu \end{cases} \tag{4.26}$$

according to intuition.
Note that he extinction is certain if $\lambda \leq \mu$ and it is still possible if $\lambda > \mu$.

Chapter 5

Fractional extensions of the Poisson process

5.1 The space-fractional Poisson process: standard case

It is well-known that the state probabilities of the homogeneous Poisson process satisfy the equations

$$\frac{dp_k(t)}{dt} = -\lambda p_k(t) + \lambda p_{k-1}(t) = -\lambda(I-B)p_k(t) \tag{5.1}$$

where $Ip_k = p_k$ and $Bp_k = p_{k-1}$ where B is the "shift" operator.
The space-fractional Poisson process is defined by considering that the state probabilities (see [14]) satisfy

$$\frac{dp_k(t)}{dt} = -\{\lambda(I-B)\}^\alpha p_k(t), \quad 0<\alpha<1 \quad k \geq 0 \tag{5.2}$$

so that we arrive at the so-called space-fractional Poisson process, denoted as $N^\alpha(t), t>0$.
Equation (5.2) can be rewritten by expanding the operator $(I-B)^\alpha$ in binomial series, because

$$(1-x)^\alpha = \sum_{h=0}^{\infty} \frac{\Gamma(\alpha+1)}{h!\Gamma(\alpha+1-h)}(-x)^h 1^{\alpha-h}, \quad |x|<1 \tag{5.3}$$

so that

$$(I-B)^\alpha = \sum_{h=0}^{\infty} \frac{\Gamma(\alpha+1)}{h!\Gamma(\alpha+1-h)}(-B)^h I^{\alpha-h} \tag{5.4}$$

and, substituting in equation (5.2)

$$\frac{dp_k(t)}{dt} = -\lambda^\alpha \sum_{h=0}^{k} \frac{\Gamma(\alpha+1)}{h!\Gamma(\alpha+1-h)}(-1)^h p_{k-h}(t) \tag{5.5}$$

subject to the same initial conditions as in (5.1), that is

$$p_{-1}(t) = 0 \quad \forall t \quad \text{and} \quad p_k(0) = \begin{cases} 1 & k=0 \\ 0 & k \geq 1 \end{cases} \tag{5.6}$$

5.1.1 Properties of the space-fractional Poisson process

(i) Note that $\frac{dp_k(t)}{dt}$ in (5.5) depends on all probabilities $p_{k-h}(t)$ with $0 \leq h \leq k$. This means that the jumps of the space-fractional Poisson process can be of any integer-valued size (cf. Fig. 5.1).

(ii) For $\alpha = 1$ equation (5.5) reduces to equation (5.1).

(iii) The distribution of jumps can be written as[1]

$$\Pr\{N^\alpha(t, t+dt] = k\} = \begin{cases} \frac{(-1)^{k+1}\lambda^\alpha}{k!}\alpha(\alpha-1)\dots(\alpha-k+1)dt + o(dt) & k \geq 1 \\ 1 - \lambda^\alpha dt + o(dt) & k = 0 \end{cases} \tag{5.7}$$

where $\alpha(\alpha-1)\dots(\alpha-k+1)$ can be represented by the Pochhammer symbol $(\alpha)_k$; alternatively the distribution (5.7) can be rewritten as

$$\begin{aligned} \Pr\{N^\alpha(t, t+dt] = k\} &= \frac{(-1)^{k+1}\lambda^\alpha\Gamma(\alpha+1)}{k!\Gamma(\alpha+1-k)} \\ &= \begin{cases} dt\frac{\lambda^k}{k!}\int_0^\infty e^{-\lambda s}s^k\nu(ds) + o(dt) & k \geq 1 \\ 1 - dt\int_0^\infty(1-e^{-\lambda s})\nu(ds) + o(dt) & k = 0 \end{cases} \end{aligned} \tag{5.8}$$

where $\nu(ds) = \frac{\alpha s^{-\alpha-1}}{\Gamma(1-\alpha)}ds, s > 0$ is the Lévy measure that is a measure on the half-line such that

$$\int_0^\infty (s \wedge 1)\,\nu(ds) < \infty \tag{5.9}$$

(iv) The probability generating function of $N^\alpha(t)$ satisfies the equation

$$\begin{cases} \frac{dG_\alpha(u,t)}{dt} &= -\lambda^\alpha(1-u)^\alpha G(u,t) \\ G_\alpha(u,0) &= 1 \end{cases} \tag{5.10}$$

so the p.g.f. is obtained by solving the (5.10):

$$G_\alpha(u,t) = e^{-t[\lambda(1-u)]^\alpha} \tag{5.11}$$

The p.g.f. (5.10) is itself a probability and can be written as:

$$G_\alpha(u,t) = e^{-t[\lambda(1-u)]^\alpha} = \Pr\left\{\min_{1 \leq k \leq N(t)} X_k^{\frac{1}{\alpha}} \geq 1-u\right\} \quad 0 < u < 1 \tag{5.12}$$

where the X_k's are independent r.v.'s and have uniform distribution in $[0,1]$, $N(t)$ is an homogeneous Poisson process of rate λ^α.

[1] In fact, by recognising that $\frac{\lambda^{k-\alpha}}{\Gamma(k-\alpha)}\int_0^\infty e^{-\lambda s}s^{k-\alpha-1}ds = 1$, it holds that $\frac{\lambda^k}{k!}\int_0^\infty e^{-\lambda s}s^k\frac{\alpha s^{-\alpha-1}}{\Gamma(1-\alpha)}ds = \frac{\lambda^k}{k!}\frac{\alpha}{\Gamma(1-\alpha)}\frac{\Gamma(k-\alpha)}{\lambda^{k-\alpha}}$. The last expression can be further manipulated as follows: $\frac{\lambda^\alpha}{k!}\frac{\alpha\Gamma(k-\alpha)}{\Gamma(1-\alpha)} = \frac{\lambda^\alpha}{k!}\frac{\alpha\Gamma(k-\alpha)}{\Gamma(1-\alpha)} \cdot \frac{\Gamma(\alpha)\cdot\Gamma(1-(k-\alpha))}{\Gamma(\alpha)\cdot\Gamma(1-(k-\alpha))} = \frac{\lambda^\alpha}{k!}\frac{\Gamma(\alpha+1)}{\Gamma(k-\alpha+1)} \cdot \frac{\Gamma(k-\alpha)\Gamma(1-\Gamma(k-\alpha))}{\Gamma(\alpha)\Gamma(1-\alpha)} = \frac{\lambda^\alpha}{k!}\frac{\Gamma(\alpha+1)}{\Gamma(k-\alpha+1)} \cdot \frac{\sin(\pi\alpha)}{\sin(\pi(k-\alpha))} = \frac{(-1)^{k+1}}{k!}\frac{\lambda^\alpha\Gamma(\alpha+1)}{\Gamma(k-\alpha+1)}$, where the Euler's "reflection formula" of the gamma function (i.e. i.e., for $0 < z < 1$, $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$) has been applied twice and by the bisection formula $\sin(\pi(k-\alpha)) = \sin(k\pi)\cos(\pi\alpha) - \cos(k\pi)\sin(\pi\alpha) = (-1)^{k+1}\sin(\pi\alpha)$.

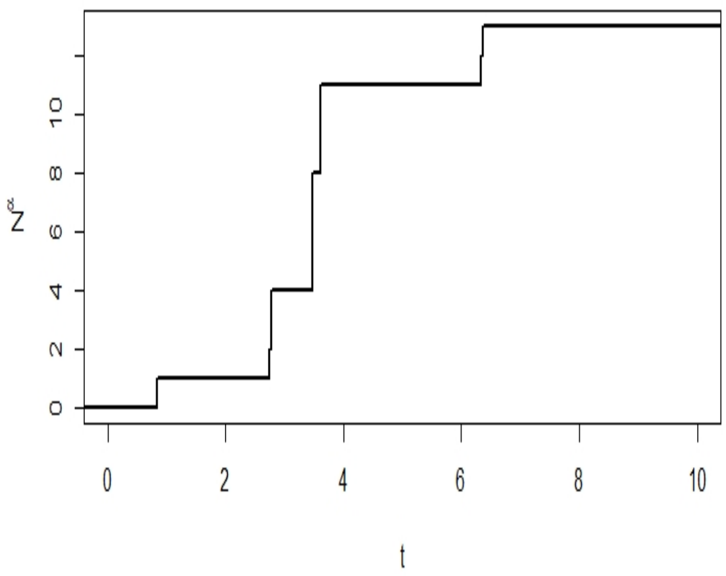


Figure 5.1: Sample path for $\alpha < 1$ of a space-fractional Poisson process $N^\alpha(t)$.

(v) By the exponential function properties in (5.11) the space-fractional process has independent increments.

In fact

$$G_\alpha(u,t) = e^{-\lambda^\alpha(1-u)^\alpha t} = e^{-\lambda^\alpha(1-u)^\alpha[s+t-s]} = G_\alpha(u,s) \cdot G_\alpha(u,t-s). \tag{5.13}$$

(vi) From (5.11) the probability law of $N^\alpha(t)$ can be obtained as:

$$p_k^\alpha(t) = Pr\left\{N^\alpha(t) = k\right\} = \frac{(-1)^k}{k!}\sum_{r=0}^{\infty}\frac{(-\lambda^\alpha t)^r}{r!}\frac{\Gamma(\alpha r+1)}{\Gamma(\alpha r+1-k)} \quad \geq 0. \tag{5.14}$$

For $\alpha = 1$ immediately emerges that (5.14) is equal to the classic Poisson distribution. Note that:

$$\begin{aligned}
p_0^\alpha(t) &= e^{-\lambda^\alpha t} \\
p_1^\alpha(t) &= \alpha\lambda^\alpha t e^{-\lambda^\alpha t} \\
p_2^\alpha(t) &= \frac{e^{-\lambda^\alpha t}}{2!}\left[(\alpha\lambda^\alpha t)^2 + \alpha(1-\alpha)\lambda^\alpha t\right] \\
p_3^\alpha(t) &= \frac{e^{-\lambda^\alpha t}}{3!}\left[(\alpha\lambda^\alpha t)^3 + 3(\alpha\lambda^\alpha t)^2(1-\alpha) + \alpha(1-\alpha)(2-\alpha)\lambda^\alpha t\right] \\
p_4^\alpha(t) &= \frac{e^{-\lambda^\alpha t}}{4!}[(\alpha\lambda^\alpha t)^4 + 6(\alpha\lambda^\alpha t)^3(1-\alpha) \\
&\quad + 6(\alpha\lambda^\alpha t)^2(1-\alpha)(2-\alpha) + \alpha(1-\alpha)(2-\alpha)(3-\alpha)\lambda^\alpha t]
\end{aligned} \tag{5.15}$$

and in general:

$$p_k^\alpha(t) = \frac{e^{-\lambda^\alpha t}}{k!}\left[c_{k,k}t^k + c_{k-1,k}t^{k-1} + \ldots + c_{2,k}t^2 + c_{1,k}t\right] \tag{5.16}$$

See [18] for expressions of the coefficients (cf. Fig. 5.2 and Fig. 5.3).

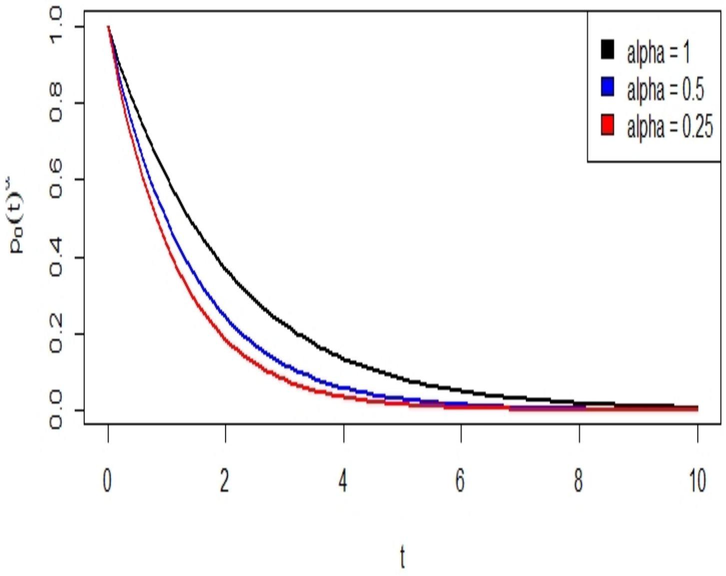


Figure 5.2: Space-fractional Poisson: The time evolution for $\alpha = 1$, $\alpha = 0.5$ and $\alpha = 0.25$ of the (survival) probability $p_0^\alpha(t)$, i.e. $k = 0$ ($\lambda = 0.5$).

(vii) Mean and variance diverge. In fact the first moment can be obtained from the first derivative of the p.g.f.

$$\begin{aligned}\frac{\partial}{\partial u}\mathbb{E}u^{N^\alpha(t)}|_{u=1} = \mathbb{E}N^\alpha(t)u^{N^\alpha(t)-1}|_{u=1} = \mathbb{E}N^\alpha(t),\\ \frac{\partial}{\partial u}G_\alpha(u,t)|_{u=1} = \frac{\partial}{\partial u}e^{-\lambda^\alpha(1-u)^\alpha t}|_{u=1}\\ G_\alpha(u,t)(\alpha\lambda^\alpha t(1-u)^{\alpha-1})|_{u=1} = \infty\end{aligned} \tag{5.17}$$

being $\alpha < 1$.

The second moment can be obtained from the k-th factorial moment

$$\mu_{(k)}^X = \mathbb{E}[X(X-1)...(X-k+1)] \tag{5.18}$$

$$\mathbb{E}[X^2] = \mu_{(1)}^X + \mu_{(2)}^X \tag{5.19}$$

and the second moment diverges as well.

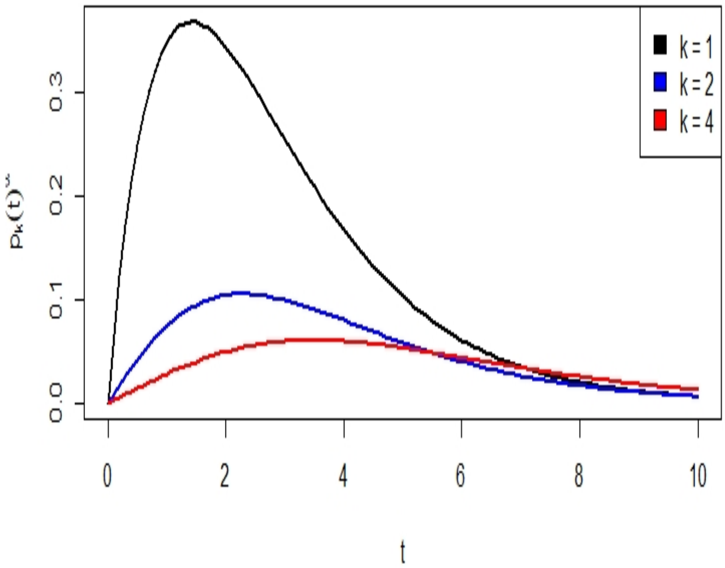


Figure 5.3: Space-fractional Poisson: The time evolution of the state probabilities $p_k^\alpha(t)$ for $k = 1$, $k = 2$ and $k = 4$, and $\alpha = 0.5$ ($\lambda = 0.5$) (cf. Fig. 5.2).

(viii) The space-fractional Poisson process is a time-changed Poisson process $N^\alpha(t)$:

$$N^\alpha(t) \overset{D}{\sim} N(S_\alpha(t)) \tag{5.20}$$

where $S_\alpha(t)$ is a stable subordinator of order $0 < \alpha < 1$, independent from $N(t)$, that is a process with Laplace transform (see Appendix B)

$$\mathbb{E}\left[e^{-\gamma S_\alpha(t)}\right] = e^{-t\gamma^\alpha} \tag{5.21}$$

It is simple to check (5.20), having in mind (5.21), as follows

$$\begin{aligned} \mathbb{E}\left[u^{N(S^\alpha(t))}\right] &= \mathbb{E}\left[E\left(u^{N(S^\alpha(t))}\middle|S^\alpha(t)\right)\right] = \mathbb{E}\left[u^{S^\alpha(t)(1-u)}\right] \\ &= e^{-t[\lambda(1-u)]^\alpha} = \mathbb{E}\left[u^{N^\alpha(t)}\right] = G_\alpha(u,t) \quad \text{ for } |u| \le 1 \end{aligned} \tag{5.22}$$

(ix) Due to the possibility for the space-fractional Poisson process of multiple jumps over any interval dt, the probabilistic nature of waiting times is different from the exponential distribution characterising the homogeneous Poisson process. The first passage[2] time of level k is defined as:

[2]Note that, in the case of the space-fractional Poisson process, the first passage time, as defined by equation (5.23), differs from the concept of *hitting time*, defined as $\hat{T}_k^{(\alpha)} = \inf\{t \ge 0 : N^{(\alpha)}(t) = k\}$, for which $\Pr\{\hat{T}_k^{(\alpha)} < \infty\} < 1$ and there is a positive probability of missing any level k (due to multiple jumps), cf. [8].

$$T_k^{(\alpha)} = \inf\{t \geq 0 : N^{(\alpha)}(t) \geq k\} \tag{5.23}$$

The distribution of $T_k^{(\alpha)}$ is obtained in [18] as

$$\Pr\{T_k^{(\alpha)} \in ds\} = -\frac{d}{ds}\sum_{h=0}^{k-1}\frac{(-\lambda)^h}{h!}\frac{d^h}{d\lambda^h}e^{-s\lambda^\alpha}, \quad s > 0 \tag{5.24}$$

where, for instance

$$\Pr\{T_k^{(\alpha)} \in ds\} = \begin{cases} \lambda^\alpha e^{-\lambda^\alpha s} ds & k = 1 \\ \lambda^\alpha e^{\lambda^\alpha s}(1-\alpha+\alpha\lambda^\alpha s)ds... & k = 2 \end{cases}. \tag{5.25}$$

[18] also derives an iterative construction, i.e.

$$\Pr\{T_k^{(\alpha)} \in ds\} = \Pr\{T_{k-1}^{(\alpha)} \in ds\} - \frac{(-\lambda)^{k-1}}{(k-1)!}\frac{d}{ds}\{\frac{d^{k-1}}{d\lambda^{k-1}}e^{-\lambda^\alpha s}\}ds. \tag{5.26}$$

Remark 5.1. In motor vehicle insurance, a tentative interpretation of subordination to a random time can be car mileage, which can represent an operative measure for the "flow of time".

Remark 5.2. The space-fractional Skellam process is defined as

$$S(t) = N_1(S_{\alpha_1}(t)) - N_2(S_{\alpha_2}(t))$$

with S_{α_1} and S_{α_2} subordinators independent of $N_1(t)$ and $N_2(t)$.
It can perform upward and downward jumps of arbitrary lengths. For its properties see [4].

5.2 Generalized space-fractional Poisson processes

If we start from equation

$$\frac{dp_k(t)}{dt} = -f\left(\lambda\left(I-B\right)\right)p_k(t) \tag{5.27}$$

which generalizes (5.1) and (5.2), we are able to construct a new class of point processes with independent increments.
The functions f appearing in (5.27) must belong to the class of Bernštein functions, which are $C^\infty(\mathbb{R}^+)$, non negative and such that[3]

$$(-1)^k\frac{d^k}{dx^k}f(x) \leq 0, \quad x > 0, k \geq 1 \tag{5.28}$$

Furthermore the Bernštein functions have the following integral representation

$$f(x) = a + bs + \int_0^\infty (1-e^{-xs})\nu(ds) \tag{5.29}$$

[3]Note that $f(x)$ is such that $f'(x) \sim \int_0^\infty e^{-xs}\nu(ds) > 0$ and $f''(x) \sim -\int_0^\infty e^{-xs}\nu(ds) < 0$.

where $\nu(ds)$ is the Lévy measure on $(0, +\infty)$ and such that

$$\int_0^{+\infty} (s \wedge 1)\nu(ds) < \infty \tag{5.30}$$

In our text we assume $a = b = 0$. For each Lévy measure we define a Bernštein function by means of (5.29).

Some particular Bernštein functions are

$$\begin{aligned}
f(x) &= x && \textit{homogeneus Poisson} \\
f(x) &= x^\alpha && \textit{standard space-fractional Poisson} \\
f(x) &= (x+\theta)^\alpha - \theta^\alpha && \textit{tempered Poisson process} \\
f(x) &= \log(1+x) && \textit{negative binomial process}
\end{aligned} \tag{5.31}$$

The function x^α is obtained by the Lévy measure

$$\nu(ds) = \frac{\alpha s^{-(\alpha+1)}}{\Gamma(1-\alpha)} ds, \quad 0 < \alpha < 1. \tag{5.32}$$

The function $(x+\theta)^\alpha - \theta^\alpha$ generates a process (tempered Poisson) with finite mean and variance (see [18]).

The process related to (5.27) is denoted by $N^f(t)$, with $t > 0$ and its probabilistic behaviour is governed by the following relationship

$$Pr\left\{N^f(t, t+dt] = k\right\} = \begin{cases} dt\frac{\lambda^k}{k!}\int_0^\infty e^{-\lambda s} s^k \nu(ds) + o(dt) & k \geq 1 \\ 1 - dt f(\lambda) & k = 0 \end{cases} \tag{5.33}$$

The probability generating function of N^f reads

$$\mathbb{E}\left[u^{N^f(t)}\right] = e^{-tf(\lambda(1-u))} \tag{5.34}$$

and can be derived by considering that

$$\begin{aligned} \frac{\partial G^f}{\partial t}(u,t) &= -f\{\lambda(I-B)\} G^f(u,t) \\ G^f(u,0) &= 1 \end{aligned} \tag{5.35}$$

The relation with the homogeneous Poisson $N(t)$ is given by

$$N^f(t) = N(H^f(t)) \tag{5.36}$$

where $H^f(t)$ is a subordinator with Laplace transform

$$\mathbb{E}\left[e^{-\mu H^f(t)}\right] = e^{-tf(\mu)} = e^{-t\int_0^\infty (1-e^{-s\mu})}\nu(ds) \tag{5.37}$$

If $f(x) = x^\alpha$, with $0 < \alpha < 1$, then $H^f(t)$ is a stable subordinator and $N^f(t)$ is the standard space-fractional Poisson process dealt with above.

If $f(x) = (x+\theta)^\alpha - \theta^\alpha$, corresponding to the Lévy measure $\nu(ds) = \frac{\alpha s^{-\alpha-1}e^{-\theta s}}{\Gamma(1-s)}ds$ with $\theta > 0, 0 < \alpha < 1$ we have the Poisson process with tempered or relativistic stable distribution. Its probability law is

$$Pr\left\{N^{\alpha,\theta}(t) = m\right\} = \frac{(-1)^m}{m!}\frac{\lambda^m e^{\theta^\alpha t}}{(\theta+\lambda)^m}\sum_{k=0}^{\infty}\frac{(-t(\lambda+\theta))^k}{k!}\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+1-m)} \quad m \geq 0 \tag{5.38}$$

If $\theta = 0$ then the (5.38) is equal to the standard case (5.14), while if $\theta = 0$ and $\alpha = 1$ (5.38) coincides with the distribution of the homogeneous Poisson process in (1.4).

Note that

$$\begin{aligned} \mathbb{E}\left[N^{\alpha,\theta}(t)\right] &= \alpha\lambda\theta^{\alpha-1}t \\ \mathbb{V}ar\left[N^{\alpha,\theta}(t)\right] &= \alpha\lambda\theta^{\alpha-2}(\lambda(1-\alpha)+\theta)t \\ Cov\left[N^{\alpha,\theta}(t), N^{\alpha,\theta}(s)\right] &= \alpha\lambda\theta^{\alpha-2}(\lambda(1-\alpha)+\theta)(s\wedge t) \end{aligned} \tag{5.39}$$

Remark 5.3. To show that $N^{1,0}(t) = N(t)$, substitute $\theta = 0$, $\alpha = 1$ in (130)

$$\begin{aligned} \lim_{\substack{\alpha\to1\\ \theta\to0}} \Pr\{N^{\alpha,\theta}(t) = m\} &= \frac{(-1)^m}{m!}\frac{\lambda^m}{\lambda^m}\sum_{k=0}^{\infty}\frac{(-\lambda t)^k}{k!}\frac{\Gamma(k+1)}{\Gamma(k+1-m)} \\ = (-1)^m\sum_{k=0}^{\infty}\frac{(-\lambda t)^k}{k!}\frac{k!}{m!(k-m)!} &= (-1)^m\sum_{k=0}^{\infty}\frac{(-\lambda t)^k}{m!(k-m)!} \quad for j = k-m \\ = (-1)^m\sum_{k=0}^{\infty}\frac{(-\lambda t)^{j+m}}{m!j!} &= \frac{(-1)^m(-\lambda t)^m}{m!}\sum_{k=0}^{\infty}\frac{(-\lambda t)^j}{j!} \\ &= e^{-\lambda t}\frac{(\lambda t)^m}{m!}. \end{aligned}$$

If $f(x) = \log(1+x)$ and the related Lévy measure is $\nu(ds) = \frac{e^{-s}}{s}ds \quad s > 0$ we have a Poisson process $N^\Gamma(t)$, $t > 0$, with the Gamma-subordinator. Its probability generating function is

$$G^\Gamma(u,t) = e^{-t\log(1+\lambda(1-u))} = \frac{1}{[1+\lambda(1-u)]^t} \tag{5.40}$$

The Γ-Poisson process has the structure of a renewal process with intertime U possessing distribution

$$Pr\{U > t\} = G^\Gamma(0,t) = (1+\lambda)^{-t} \tag{5.41}$$

The process $N^\Gamma(t) \overset{D}{\sim} N[H^\Gamma(t)]$ has Laplace transform

$$\mathbb{E}\left[e^{-\mu H^f(t)}\right] = (1+\mu)^{-t} \tag{5.42}$$

while the probability distribution of $N^\Gamma(t)$ reads

$$Pr\left\{N^\Gamma(t) = k\right\} = \frac{\lambda^k\Gamma(k+t)}{\Gamma(t)k!(\lambda+1)^{k+t}} \quad k \geq 0 \tag{5.43}$$

and has the form of a negative binomial distribution, i.e.

$$Pr\left\{B^i = k\right\} = \frac{\Gamma(i+k)}{\Gamma(i)\Gamma(k+i)} p^i q^k = \binom{k+i-1}{k} p^i q^k \quad k \geq 0, i \geq 1. \tag{5.44}$$

With $p = (1+\lambda)^{-1}$, $q = \lambda(1+\lambda)^{-1}$ and $i = t$ then equation (5.44) is equal to (5.43). The distributions of jumps in this case is

$$Pr\left\{N^\Gamma(t, t+dt] = k\right\} = \begin{cases} \left(\frac{\lambda}{\lambda+1}\right)^k \frac{1}{k} dt & k \geq 1 \\ 1 - \log(1+\lambda)dt & k = 0 \end{cases} \tag{5.45}$$

which is a simil "logarithmic distribution". The main moments of $N^\Gamma(t)$ are

$$\begin{aligned} \mathbb{E}\left[N^\Gamma(t)\right] &= \lambda t \\ \mathbb{V}ar\left[N^\Gamma(t)\right] &= \lambda(\lambda+1)t \\ Cov\left[N^\Gamma(s), N^\Gamma(t)\right] &= \lambda(\lambda+1)(s \wedge t). \end{aligned} \tag{5.46}$$

Note that

$$\begin{aligned} Pr\left\{N^\Gamma(s) = r | N^\Gamma(t) = k\right\} &= \tbinom{k}{r} \tfrac{B(s+r, t-s+k-r)}{B(s, t-s)} = \\ &= E\left[\tbinom{k}{r} X^r (1-X)^{k-r}\right] \end{aligned} \tag{5.47}$$

where X is a Beta random variable with parameters s and $t-s$, that is

$$Pr\left\{X \in dx\right\} = \frac{x^{s-1}(1-x)^{t-s-1}}{B(s, t-s)} dx, \quad 0 < x < 1 \tag{5.48}$$

with $B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$[4].

5.3 Another fractional generalization of the Poisson process

A fractional Poisson process $\hat{N}_\nu(t)$ can be defined by the distribution (see [2])

$$Pr\left\{\hat{N}_\nu(t) = k\right\} = \frac{1}{E_{\nu,1}(\lambda t)} \frac{(\lambda t)^k}{\Gamma(\nu k + 1)}, \qquad k \geq 0 \tag{5.49}$$

where

$$E_{\nu,\mu}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\nu k + \mu)} \tag{5.50}$$

is the Mittag-Leffler function, with $\nu, \mu > 0$ The probability generating function of (5.49) is

$$G_\nu(u,t) = \mathbb{E}\left[u^{\hat{N}_\nu(t)}\right] = \frac{E_{\nu,1}(\lambda u t)}{E_{\nu,1}(\lambda t)} \qquad 0 \leq u \leq 1 \tag{5.51}$$

which satisfies a order-ν fractional differential equation in u (see Appendix C):

$$\begin{cases} \frac{\partial^\nu G_\nu(u,t)}{\partial u^\nu} = \lambda t G_\nu(u,t) & 0 < \nu < 1 \\ G_\nu(u,0) = 1 \end{cases} \tag{5.52}$$

[4]Note that the result generalizes the property of the Poisson process reported in equation (1.31).

The expected value of $\hat{N}_\nu(t)$ is, as can be ascertained using $\frac{d}{du}E_{\nu,1}(\lambda ut) = \frac{\lambda t}{\nu}E_{\nu,\nu}(\lambda ut)$,

$$\mathbb{E}\left[\hat{N}_\nu(t)\right] = \frac{\lambda t}{\nu}\frac{E_{\nu,\nu}(\lambda t)}{E_{\nu,1}(\lambda t)} \tag{5.53}$$

while the variance is:

$$\mathbb{V}ar\left[\hat{N}_\nu(t)\right] = \frac{\lambda t}{\nu^2}\frac{E_{\nu,\nu-1}(\lambda t)}{E_{\nu,1}(\lambda t)} + \frac{\lambda t}{\nu^2}\frac{E_{\nu,\nu}(\lambda t)}{E_{\nu,1}(\lambda t)}\left\{1 - \lambda t\frac{E_{\nu,\nu}(\lambda t)}{E_{\nu,1}(\lambda t)}\right\} \tag{5.54}$$

For $\nu = 1$, $E_{1,1}(x) = e^x$ and $\hat{N}_1(t)$ is the simple Poisson process. For $\nu < 1$, $\hat{N}_\nu(t)$ has no longer independent increments.

Remark 5.4. This process can be viewed as a fractional version of the Poisson process, because the Mittag-Leffler function has a similar role as the exponential function e^x in the analysis of equations with fractional order derivatives.

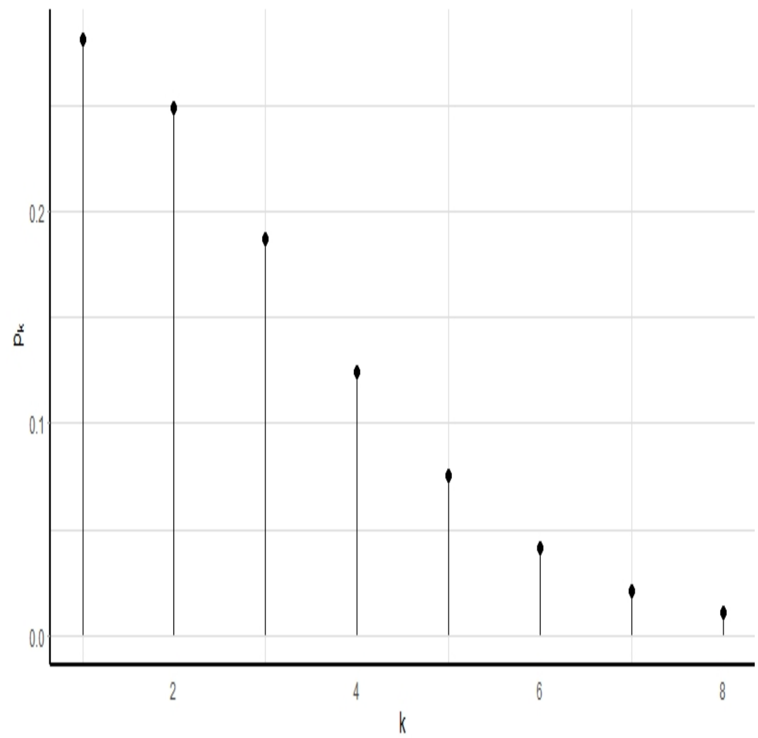


Figure 5.4: Distribution $p_k(t), k \geq 1$ for $\lambda = 1$ and $\nu = 0.5$.

It is easy to show that for i.i.d. r.v.'s with d.f. $F(x)$[5]:

$$Pr\left\{\max\left(X_1, \ldots, X_{\hat{N}_\nu(t)}\right) < u\right\} = \frac{E_{\nu,1}(\lambda t F(u))}{E_{\nu,1}(\lambda t)}. \tag{5.55}$$

[5]Cf. the Gnedenko-Gumbel distribution in formula (1.72)

In fact

$$\begin{aligned} &\Pr\{\max\{X_1,...,X_{\hat{N}_\nu(t)}\}<u\} \\ &=\mathbb{E}_{\hat{N}_\nu}[\Pr\{\max\{X_1,...,X_{\hat{N}_\nu(t)}\}<u|\hat{N}_\nu(t)\}] \\ =\mathbb{E}_{\hat{N}_\nu}[\Pr(X_1<u)^{\hat{N}_\nu(t)}] = \sum_{k=0}^{\infty}\Pr(X_1<u)^k &\frac{(\lambda t)^k}{\Gamma(\nu k+1)}\frac{1}{E_{\nu,1}(\lambda t)} = \frac{E_{\nu,1}(\lambda t F(u))}{E_{\nu,1}(\lambda t)}. \end{aligned} \tag{5.56}$$

Moreover,

$$\Pr\{\min(X_1,...,X_{\hat{N}_\nu(t)})>u\} = \frac{E_{\nu,1}(\lambda t(1-F(u)))}{E_{\nu,1}(\lambda t)} \tag{5.57}$$

The distribution of the waiting time of the first event T_1^ν is

$$\Pr\{\hat{T}_1^\nu>t\} = \Pr(\hat{N}_\nu(t)=0) = \frac{1}{E_{\nu,1}(\lambda t)}. \tag{5.58}$$

Remark 5.5. The distribution (5.49) can be seen as a weighted transformation of the homogeneous Poisson process (see Fig. 5.4):

$$Pr\left\{\hat{N}_\nu(t)=k\right\} = \frac{Pr\{N(t)=k\}\frac{k!}{\Gamma(\nu k+1)}}{\sum_{j=0}^{\infty}Pr\{N(t)=j\}\frac{j!}{\Gamma(\nu j+1)}} \quad k\geq 0$$

.

5.4 The time-fractional Poisson process

In this case it is useful to start the analysis from the equation governing the state probabilities (see [2]) generalized by a (time)-fractional derivative

$$\frac{d^\nu p_k(t)}{dt^\nu} = -\lambda p_k(t)+\lambda p_{k-1}(t) \quad k\geq 0, p_{-1}=0, 0<\nu<1 \tag{5.59}$$

subject to the initial conditions

$$p_k(0) = \begin{cases} 1 & k=0, \\ 0 & k\geq 1, \end{cases} \tag{5.60}$$

where

$$\frac{d^\nu p_k(t)}{dt^\nu} = \frac{1}{\Gamma(1-\nu)}\int_0^t \frac{dp_k(s)}{ds}\frac{1}{(t-s)^\nu}\,ds \tag{5.61}$$

is the fractional derivative in the sense of Dzerbayshan-Caputo.
It can be shown (see [2]) that the distribution emerging from (5.59) has the form

$$\begin{aligned} p_k^\nu(t) &= \frac{(\lambda t^\nu)^k}{k!}\sum_{r=0}^{\infty}\frac{(r+k)!}{r!}\frac{(-\lambda t^\nu)^r}{\Gamma(\nu(k+r)+1)} \\ &= \sum_{r=k}^{\infty}(-1)^{r-k}\binom{r}{k}\frac{(\lambda t^\nu)^r}{\Gamma(\nu r+1)} \qquad k\geq 0 \end{aligned} \tag{5.62}$$

and the process can be called time-fractional Poisson process $N_\nu(t)$.
For small values of k it is possible to write the distribution (5.62) in terms of Mittag-Leffler functions

as follows

$$\begin{cases} p_0^\nu(t) = E_{\nu,1}(-\lambda t^\nu) \\ p_1^\nu(t) = \frac{\lambda t^\nu}{\nu} E_{\nu,\nu}(-\lambda t^\nu) \\ p_2^\nu(t) = \frac{(\lambda t^\nu)^2}{2!\nu^2} \left[(1-\nu)E_{\nu,2\nu}(-\lambda t^\nu) + E_{\nu,2\nu-1}(-\lambda t^\nu)\right] \\ p_3^\nu(t) = \frac{(\lambda t^\nu)^3}{3!\nu^3} \left[2(1-\nu)\left(\frac{1}{2}-\nu\right) E_{\nu,3\nu}(-\lambda t^\nu) + 3(1-\nu)E_{\nu,3\nu-1}(-\lambda t^\nu) + E_{\nu,3\nu-2}(-\lambda t^\nu)\right] \end{cases} \tag{5.63}$$

The probability generating function of the time fractional Poisson process $N_\nu(t)$ is

$$G_{N_\nu}(u) = E_{\nu,1}\left(\lambda(u-1)t^\nu\right) \tag{5.64}$$

From (5.64) we can extract the mean and the variance (cf. Fig. 5.5 and Fig 5.6)

$$\begin{aligned} \mathbb{E}\left[N_\nu(t)\right] &= \frac{\lambda t^\nu}{\Gamma(\nu+1)} \\ \mathbb{V}ar\left[N_\nu(t)\right] &= \frac{\lambda t^\nu}{\Gamma(\nu+1)} + \frac{(\lambda t^\nu)^2}{\nu}\left\{\frac{1}{\Gamma(2\nu)} - \frac{1}{\nu\Gamma^2(\nu)}\right\} \end{aligned} \tag{5.65}$$

Clearly, the time-fractional Poisson process is overdispersed[6].

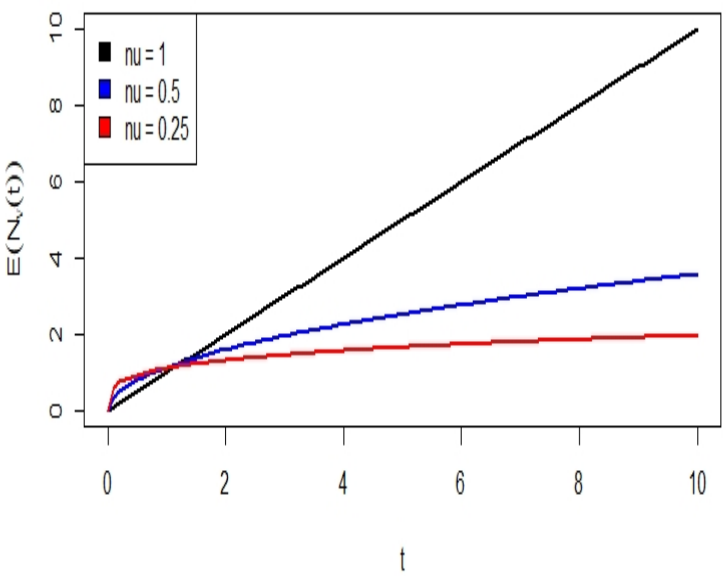


Figure 5.5: Comparison between expected values (w.r.t. time) of a time-fractional Poisson process of different parameters $\nu = \{1, 0.5, 0.25\}$ and $\lambda = 1$. Note: $\nu = 1$ corresponds to the standard Poisson process.

The time-fractional Poisson process is a renewal process with intertimes $U_j = \tau_j - \tau_{j-1}$ having the Mittag-Leffler distribution

$$Pr\left\{U_j > t\right\} = \Pr\{T_1^\nu > t\} = Pr\left\{N_\nu(t) = 0\right\} = E_{\nu,1}(-\lambda t^\nu) = p_0^\nu(t) \tag{5.66}$$

[6]For $0 < \nu < 1$, the term $\Delta(\nu) = \frac{1}{\Gamma(2\nu)} - \frac{1}{\nu\Gamma^2(\nu)} > 0$. In fact, by means of the Legendre duplication formula of the gamma function, i.e. $\Gamma(2z) = \frac{\Gamma(z)\Gamma(z+\frac{1}{2})}{2^{1-2z}\sqrt{\pi}}$, one can easily check that $\Delta(\nu) > 0$, $0 < \nu < 1$.

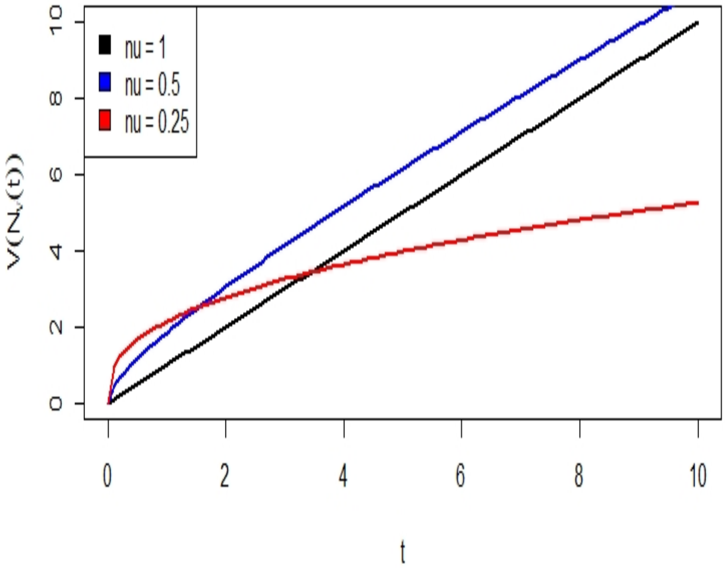


Figure 5.6: Comparison between variances (w.r.t. time) of a time-fractional Poisson process of different parameters $\nu = \{1, 0.5, 0.25\}$ $(\lambda = 1)$. Note: $\nu = 1$ corresponds to the standard Poisson process.

One can notice (cf. Fig 5.7) how intertimes have longer durations than exponential, corresponding to "plateau" with no jumps, followed by clusters of (unit) jumps. Indeed the intertimes have infinite expectations.
The density of the random variables U_j is

$$Pr\{U_j \in dt\} = \lambda t^{\nu-1} E_{\nu,\nu}(-\lambda t^{\nu})dt \quad t > 0, 0 < \nu < 1 \tag{5.67}$$

and has Laplace transform

$$\int_0^{\infty} e^{-\mu t} Pr\{U_j \in dt\} = \frac{\lambda}{\lambda + \mu^{\nu}}, \quad \mu > 0. \tag{5.68}$$

We are able to calculate:

$$\begin{aligned} Pr\{N_{\nu}(t) = k\} &= Pr\{U_1 + \ldots + U_k < t, U_1 + \ldots + U_{k+1} > t\} \\ &= Pr\{U_1 + \ldots + U_k < t\} - Pr\{U_1 + \ldots + U_{k+1} < t\} \\ &= Pr\{T_k^{(\nu)} < t\} - \Pr\{T_{k+1}^{(\nu)} < t\} \end{aligned} \tag{5.69}$$

and by means of Laplace transform:

$$\int_0^{\infty} e^{-\mu t} Pr\{N_{\nu}(t) = k\} dt = \frac{\mu^{\nu-1}\lambda^k}{(\lambda + \mu^{\nu})^{k+1}} \tag{5.70}$$

that is equal to the Laplace transform of the solution.
The time-fractional Poisson process can be represented as a Poisson process with a time change

obtained by the "inverse" of a Stable subordinator (see [15]). More precisely, it admits the following representation:

$$N_\nu(t) \stackrel{D}{=} N(L^\nu(t)), \qquad 0 < \nu \leq 1, \qquad t \geq 0 \tag{5.71}$$

where N is a homogeneous Poisson process and $L^\nu(t)$ is the inverse of a Stable subordinator $S^\nu(t)$, i.e.

$$Pr\{L^\nu(t) > s\} = Pr\{S^\nu(s) < t\} \tag{5.72}$$

For the hitting time of the time-fractional Poisson process

$$\tau_k^{(\nu)} = \inf(t > 0 : N_\nu(t) = k), \quad k \geq 1, \nu \in (0, 1] \tag{5.73}$$

[2] provide the expression

$$\Pr\{\tau_k^{(\nu)} \in ds\} = \lambda_\beta^k \sum_{h=0}^{\infty} \binom{-k}{h} \lambda_\beta^h \frac{s^{\nu(k+h)-1}}{\Gamma(\nu(k+h))} ds. \tag{5.74}$$

Remark 5.6. The time-fractional Skellam process
The time-fractional Skellam process is expressed in terms of the difference of two time-fractional Poisson processes (see [4])

.

$$f_{Sk}(t) = N_1(L^{(1)}(t)) - N_2(L^{(2)}(t))$$

where $L^{(1)}(t)$ and $L^{(2)}(t)$ are two independent inverse stable subordinators of indexes ν_1 and ν_2 both in $(0, 1)$.

5.5 The space-time fractional Poisson process

A space-time fractional Poisson process N_ν^α, $\alpha, \nu \in (0, 1)$, can be easily understood by means of subordination of an ordinary Poisson process to both a Stable and *then* its inverse Stable subordinator. As such, it can be represented as $N_\nu^\alpha(t) = N(S_\alpha(L_\alpha(t)))$, $t \geq 0$, where $D_\alpha(t)$ is an independent α-Stable subordinator and $L_\alpha(t)$ is its inverse.

We refer to [14], where the process is defined as an extension to the space-fractional Poisson (see also [15]).

Quite naturally, on one side, the "space-fractional" dimension of the process involves the fractional "shift" operator $\Delta^\alpha = (1-B)^\alpha$ for its construction, as it appears in the (fractional) Cauchy problem for the state probabilities $p_k^{\alpha,\nu}(t)$, $k \geq 0$, i.e.

$$\frac{d^\nu}{dt^\nu} p_k^{\alpha,\nu}(t) = -\lambda^\alpha (1 - B)^\alpha p_k^{\alpha,\nu}(t) \tag{5.75}$$

where, on the other side, a (Caputo) time-fractional derivative operator takes into accounts the

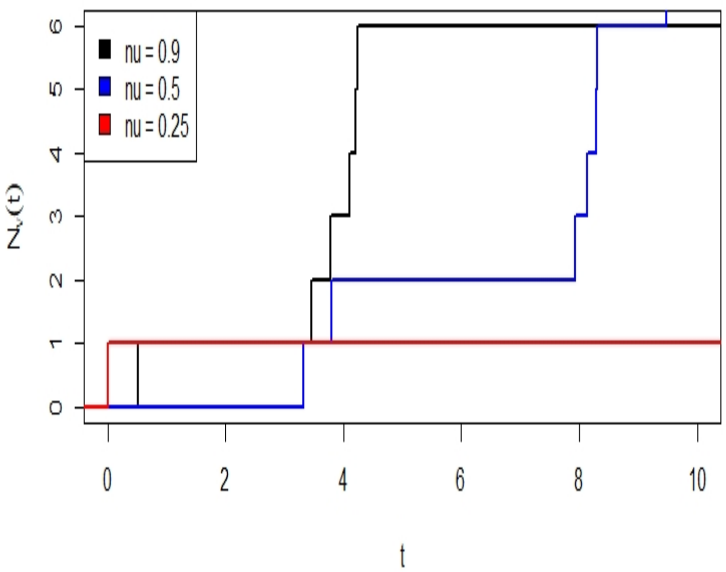


Figure 5.7: Time-fractional Poisson: A sample trajectory for $\nu = 0.9$, $\nu = 0.5$ and $\nu = 0.25$. Note the "slow pace" of progress of the counts w.r.t. time.

time-fractional dimension, subject to the initial condition

$$p_k^{\alpha,\nu}(0) = \begin{cases} 0 & k \geq 1 \\ 1 & k = 0 \end{cases} \tag{5.76}$$

The "time-fractional" and "space-fractional" dimensions appear more evident in state probabilities $p_k^{\alpha,\nu}(t) = \Pr\{N_\nu^\alpha(t) = k\}$, which are given as (see [14])

$$p_k^{\alpha,\nu}(t) = \frac{(-1)^k}{k!} \sum_{h=0}^{\infty} \frac{(-\lambda^\alpha t^\nu)^h}{\Gamma(\nu h + 1)} \frac{\Gamma(\alpha h + 1)}{\Gamma(\alpha h + 1 - k)} \qquad k \geq 1, \alpha \in (0,1], \nu \in (0,1], \tag{5.77}$$

where state-probabilities appear as a "natural" integration of the respective specific structures. Indeed, setting $\alpha = 1$ and $\nu = 1$ the time-fractional and space-fractional Poisson processes are recovered, respectively. See also [3].
The probability generating function $G_{\alpha,\nu}(u)$ obtains as solution of

$$\begin{cases} \frac{d^\nu}{dt^\nu} G_{\alpha,\nu}(u,t) = -\lambda^\alpha (1-u)^\alpha G_{\alpha,\nu}(u,t) \\ G_{\alpha,\nu}(u,0) = 1 \end{cases} \tag{5.78}$$

that is

$$G_{\alpha,\nu}(u) = E_{\nu,1}(-\lambda^\alpha (1-u)^\alpha t^\nu) \qquad |u| \leq 1 \tag{5.79}$$

where the $E_{\nu,1}(x) = E_\nu(x)$ is indeed a one-parameter Mittag-Leffler function.
The p.g.f. $G_{\alpha,\nu}(u)$ has a probabilistic interpretation with respect to the event $(\min_{1 \leq k \leq N_\nu(t)} U_k^{\frac{1}{\alpha}} \geq$

$1-u$), where the random variables U_k, $k \geq 1$, are i.i.d uniforms, and $N_\nu(t)$ is the time-fractional Poisson process, i.e.

$$G_{\alpha,\nu}(u) = \Pr\{\min_{1\leq k\leq N_\nu(t)} U_k^{\frac{1}{\alpha}} \geq 1-u\} \qquad |u| < 1. \tag{5.80}$$

Appendix A

Probability generating functions.

The probability generating function (p.g.f.) of a discrete non negative r.v. $X : \{k, p_k; k = 0, 1, 2, ...\}$ is defined as

$$G_X(u) = \sum_{k=0}^{\infty} p_k u^k \quad |u| \leq 1 \qquad \text{for absolute convergence} \tag{A.1}$$

The name comes from the property

$$\frac{1}{k!}\frac{\partial^k G_X(u)}{\partial u^k}|_{u=0} = p_k \tag{A.2}$$

Consequently, if $G_X(u) = G_Y(u)$ then $p_X = p_Y$, i.e. if X and Y have the same p.g.f. then they have the same distribution.
Note that

$$G_X(1) = \sum_{k=0}^{\infty} p_k = 1 G_X^{(1)}(1) = \frac{\partial G_X(u)}{\partial u}_{|u=1} = \mathbb{E}[X] \tag{A.3}$$

The k-th factorial moment is

$$\mu_{(k)} \equiv \mathbb{E}[X(X-1)...(X-k+1)] = \mathbb{E}[\frac{X!}{(X-k)!}] = \frac{\partial^k}{\partial u^k} G_X(u)_{|u=1} = G_X^{(k)}(1) \tag{A.4}$$

Therefore, the variance is

$$\mathbb{V}ar[X] = \mathbb{E}[X(X-1)] + \mathbb{E}[X] - \mathbb{E}^2[X] = \mu_{(2)}^X + \mu_{(1)}^X - (\mu_{(1)}^X)^2 = G_X^{(2)}(1) + G_X^{(1)}(1) - (G_X^{(1)}(1))^2. \tag{A.5}$$

In general, the p.g.f. of a r.v. X is defined as

$$G_X(u) = \mathbb{E}[u^X] \quad |u| \leq 1 \tag{A.6}$$

Given that the moment generating function (m.g.f.) is

$$M_X(\theta) \equiv \mathbb{E}[e^{\theta X}] \tag{A.7}$$

then

$$M_X(\theta) = G_X(e^{\theta}) \tag{A.8}$$

If X_i, $i = 1, 2, ..., n$, are n independent r.v.'s and $S_n = \sum_{i=1}^{n} a_i X_i$, a_i constants, then

$$G_{S_n}(u) = \mathbb{E}[u^{S_n}] = \mathbb{E}[u^{\sum_i a_i X_i}] = \mathbb{E}[u^{a_1 X_1} ... u^{a_n X_n}] = \prod_{i=1}^{n} \mathbb{E}[u^{a_i X_i}] = \prod_{i=1}^{n} G_{X_i}(u^{a_i}) \tag{A.9}$$

For the special case $S = X_1 - X_2$

$$G_S(u) = G_{X_1}(u) G_{X_2}(\frac{1}{u}) \tag{A.10}$$

If N is a discrete r.v. independent of X_i and $S_N = \sum_{i=1}^{N} X_i$

$$G_{S_N}(u) = \mathbb{E}[u^{\sum_i X_i}] = \mathbb{E}_N[\mathbb{E}[u^{\sum_{i=1} X_i} | N]] \tag{A.11}$$

$$= \mathbb{E}_N[\prod_{i=1}^{N} \mathbb{E}[u^{X_i}]] = \mathbb{E}_N[\prod_{i=1}^{N} G_{X_i}(u)] = \sum_{n=1}^{\infty} p_n \prod_{i=1}^{n} G_{X_i}(u) \tag{A.12}$$

and if the X_i are also identically distributed (i.i.d.) then

$$G_{S_N(u)} = \mathbb{E}_N[G_X(u)^N] = G_N(G_X(u)). \tag{A.13}$$

Examples

- The p.g.f. of a constant c is

$$G_c(u) = u^c. \tag{A.14}$$

- The p.g.f. of a Bernoulli r.v. X with parameter p is

$$G_X(u) = \mathbb{E}[u^X] = [pu^1 + (1-p)u^0] = [up + (1-p)]; \tag{A.15}$$

- The p.g.f. of a Binomial r.v. $S_n = \sum_{i=1}^{n} X_i$ is

$$\mathbb{E}[u^{S_n}] = \mathbb{E}[\prod_{i=1}^{n} u^{X_i}] = \prod_{i=1}^{n} \mathbb{E}[u^{X_i}] = \mathbb{E}[u^{X_1}]^n \tag{A.16}$$

respectively by independence and equi-distribution of the X_i so that finally

$$= [u^1 p + (1-p)u^0]^n = [up + (1-p)]^n \tag{A.17}$$

where p is the parameter of the Bernoulli r.v. X_i.

- The p.g.f. of a Poisson(λ) r.v. is

$$G_X(u) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} u^k = e^{\lambda(u-1)}. \tag{A.18}$$

Appendix B

Stable random variables and stable subordinators.

B.1 Stable random variables

A random variable X is Stable iff $\forall\ a_1,\ a_2 \in \mathrm{R}^+$ and $\forall X_1,\ X_2,\ X$, i.i.d., $\exists b > 0$ and $c \in \mathbb{R}$ such that:

$$a_1X_1 + a_2X_2 \overset{\mathrm{D}}{\sim} bX + c \tag{B.1}$$

If X is Stable, $\exists \alpha \in (0, 2]$ called stability index, such that:

$$a_1^\alpha + a_2^\alpha = b^\alpha. \tag{B.2}$$

We write X is α-Stable.
The c.f. of an α-Stable r.v. X is the following:

$$\mathbb{E}(e^{i\theta X}) = \begin{cases} e^{-\sigma^\alpha|\theta|^\alpha(1-i\beta\theta\tan\frac{\alpha\pi}{2})+i\mu\theta} & \alpha \neq 1 \\ e^{-\sigma|\theta|(1+i\beta\frac{2}{\pi}\theta\ln|\theta|)+i\mu\theta} & \alpha = 1 \end{cases} \tag{B.3}$$

with $\sigma > 0$, $\beta \in [-1, 1]$, $\mu \in \mathbb{R}$. We write also $X \sim S_\alpha(\sigma, \beta, \mu)$.
If $X > 0$ we have also the Laplace transform

$$\mathbb{E}(e^{-\gamma X}) = \begin{cases} \exp\{-\frac{\sigma^\alpha\gamma^\alpha}{\cos\{\frac{\alpha\pi}{2}\}}\} & \alpha \neq 1 \\ \exp\{-\sigma\frac{2}{\pi}\ln\gamma\} & \alpha = 1 \end{cases} \tag{B.4}$$

Example. Let $X \sim N(\mu, \sigma^2)$, then $\alpha = 2$ and

$$b = \sqrt{a_1^2 + a_2^2}, \tag{B.5}$$

$$c = (a_1 + a_2 - b)\mu. \tag{B.6}$$

B.2 Stable subordinators

The general idea of subordination refers to a process which evolves in an "operative" (as opposed to natural) random time.

- A subordinator is an (a.s.) non-decreasing right-continuous process with stationary, i.i.d. increments (a case of Levy process). A Stable subordinator is an α-Stable process $S_\alpha(t)$; an inverse Stable subordinator $L(x)$ represents the first passage time of a Stable subordinator S_α for a level x, i.e. $L(x) = \inf\{t : S_\alpha(t) \geq x\}$, $x > 0$. The following results hold.
- α-Stable subordinators $S_\alpha(t)$, $0 < \alpha < 1$, are characterized by Laplace transform of the form

$$\mathbb{E}[e^{-\theta S_\alpha(t)}] = e^{-\theta^\alpha t} \qquad \theta > 0, t > 0 \tag{B.7}$$

- α-Stable subordinators, $0 < \alpha < 1$, scale in time according to a power-law with exponent greater than unity, i.e.

$$S_\alpha(ct) \sim c^{\frac{1}{\alpha}} S_\alpha(t), \qquad c > 0 \tag{B.8}$$

  Inverse α-Stable subordinators, $0 < \alpha < 1$, scale in time according to a power-law with exponent smaller than unity, i.e.

$$S_\alpha(ct) \overset{d}{\sim} c^\alpha S_\alpha(t), \qquad c > 0 \tag{B.9}$$

  The density function of an $\alpha-$Stable subordinator is given by

$$h_\alpha(x,t) = \alpha t x^{-(\alpha+1)} W_\alpha(t x^{-\alpha}) \qquad x > 0 \tag{B.10}$$

  while the density function of its inverse $\alpha-$Stable subordinator is given by

$$l_\alpha(t,x) = t^{-\alpha} W_\alpha(t^{-\alpha} x) \qquad x > 0 \tag{B.11}$$

  where $W_\alpha(x)$ is the Wright function

$$W_\alpha(x) \equiv W_{-\alpha,1-\alpha}(x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!\Gamma(-\alpha k + (1-\alpha))}. \tag{B.12}$$

  The two densities are related as

$$l_\alpha(t,x) = -\frac{\partial}{\partial t} \int_0^x h_\alpha(y,t) dy \tag{B.13}$$

  Closed form densities arise for $\alpha = \frac{1}{2}$, in which case

$$h_{\frac{1}{2}}(x,t) = \frac{t}{\sqrt{4\pi x^3}} e^{-\frac{t^2}{4x}} \tag{B.14}$$

which corresponds to the first passage time distribution (Inverse Gaussian) of a standard Brownian motion through the *level* $\frac{t}{\sqrt{2}}$ (or, equivalently, of a Brownian motion with variance $\sqrt{2}$ through a level t), and

$$l_{\frac{1}{2}}(t,x) = \frac{1}{\sqrt{\pi x}} e^{-\frac{t^2}{4x}} \tag{B.15}$$

which corresponds to the probability distribution of the maximum of a Brownian motion over a *time* interval $(0, x)$.

Appendix C

Fractional derivatives.

The Riemann-Liouville fractional integral of order α is defined as

$$(I^{\alpha}f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-y)^{\alpha-1} f(y)dy, \quad \alpha > 1 \tag{C.1}$$

The fractional derivative of order α of a (well behaved) function is the ordinary m-th derivative D^m of a fractional Riemann-Liouville integral of order $m-\alpha$, i.e.

$$(D^{\alpha}f)(x) = [D^m(I^{m-\alpha}f)](x), \quad \text{for } m-1 < \alpha < m, \alpha > -1 \tag{C.2}$$

In fact the Riemann-Liouville fractional derivative of a function $f(x)$ is defined explicitly as

$$(D^{\alpha}f)(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_0^x \frac{f(y)}{(x-y)^{\alpha+1-m}} dy & m-1 < \alpha < m \\ \frac{d^m}{dx^m} f(x) & \alpha = m \end{cases} \tag{C.3}$$

Note that the fractional integral is an extension of the Cauchy formula:

$$\int_0^x dx_1 \int_0^{x_1} dx_2 ... \int_0^{x_{n-1}} f(x_n)dx_n = \frac{1}{(n-1)!} \int_0^x f(x_n)(x-x_n)^{n-1} dx_n. \tag{C.4}$$

and the m-th (ordinary) derivative of a power function x^n is

$$D^m x^n = \frac{d^m x^n}{dx^m} = n(n-1)(n-m+1)x^{n-m} = \frac{n!}{(n-m)!} x^{n-m} = \frac{\Gamma(n+1)}{\Gamma(n-m+1)} x^{n-m}, \quad m \leq n. \tag{C.5}$$

By analogy the fractional α-th derivative can be obtained directly as

$$\frac{d^{\alpha}}{dx^{\alpha}} x^n = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha}, \alpha \leq n. \tag{C.6}$$

For example the derivative of half order (i.e., $\alpha = \frac{1}{2}$) for $n = 1$ gives

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} x = \frac{\Gamma(2)}{\Gamma(\frac{3}{2})} x^{\frac{1}{2}} = \frac{x^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} = \frac{1}{\frac{1}{2}\sqrt{\pi}} x^{\frac{1}{2}}. \tag{C.7}$$

The Dzherbashyan-Caputo fractional derivative is defined as

$$(D_c^\alpha f)(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{\frac{d^m}{dy^m} f(y)}{(x-y)^{\alpha+1-m}} dy & m-1 < \alpha < m \\ \frac{d^m}{dx^m} f(x) & \alpha = m \end{cases} \tag{C.8}$$

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